TOPICAL REVIEW

The Tensor-Vector-Scalar theory and its cosmology

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Abstract. Over the last few decades, astronomers and cosmologists have accumulated vast amounts of data clearly demonstrating that our current theories of fundamental particles and of gravity are inadequate to explain the observed discrepancy between the dynamics and the distribution of the visible matter in the Universe. The Modified Newtonian Dynamics (MOND) proposal aims at solving the problem by postulating that Newton's second law of motion is modified for accelerations smaller than $\sim 10^{-10} m/s^2$. This simple amendment, has had tremendous success in explaining galactic rotation curves. However, being non-relativistic, it cannot make firm predictions for cosmology.

A relativistic theory called Tensor-Vector-Scalar (TeVeS) has been proposed by Bekenstein building on earlier work of Sanders which has a MOND limit for non-relativistic systems. In this article I give a short introduction to TeVeS theory and focus on its predictions for cosmology as well as some non-cosmological studies

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1. Introduction

Over the last few decades, astronomers and cosmologists have accumulated vast amounts of data clearly demonstrating that our current theories of fundamental particles and of gravity are inadequate to explain the observed discrepancy between the dynamics and the distribution of the visible matter in the Universe [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. This has been called the "missing mass problem" or the "mass-discrepancy" problem.

On galactic and cosmological scales, gravity is the dominant force which drives the dynamics of all matter. Our current well accepted theory of gravity is Einstein's General Relativity (GR) which explains how the dynamics of the various matter species are driven by their collective energy density and pressure. Although GR has been vigorously tested in our solar system, on cosmological scales and curvatures it has traditionally been assumed from the outset. Yet, it is on these scales where the observed dynamics fail to match the observed matter distribution, from the scales of galaxies to the scales of the Cosmic Microwave Background (CMB) radiation.

One could imagine that the missing mass is composed of baryons in objects other than stars, for example Jupiter size planets or brown dwarves, collectively called MACHOS, or baryonic dark matter. These objects cannot be seen because they do not emit light of their own. However microlensing studies did not detect the abundance needed for these objects to make up for the missing mass [11, 12, 13, 14]. Moreover the abundances of elements predicted by Big Bang Nucleosynthesis (BBN) give a matter density far below the needed mass density [15, 16].

The traditional way to explain the discrepancy between dynamics and matter distribution, is to posit a new form of matter, non-baryonic in nature, named Dark Matter. Dark Matter is thought not to interact with electromagnetic radiation and therefore cannot be detected by observing photons at various frequencies. Even though it cannot be seen directly, its presence is evident from the pull of gravity. Thus one attributes the extra gravitational force observed, to a "dark matter" component whose abundance is required to greatly exceed the visible matter abundance. Dark matter candidates have been traditionally split [17] into "hot dark matter" and "cold dark matter", although an intermediate possibility, namely "warm dark matter" is sometimes being considered.

The earliest possibility considered for a dark matter candidate was a massive neutrino [18, 19, 20, 21], since neutrinos are particles which are known to exist as well as being very weakly interacting. However, massive neutrinos cannot be the dominant form of the dark matter. If the dark matter is composed of massive neutrinos then their mass must be at most 30-70eV for reasonable values of the Hubble constant, if they are not to overclose the universe [18]. On the other hand the Tremaine-Gunn inequality [22] gives a lower bound on the neutrino mass if neutrinos are to be bounded gravitationally within some radius. For example for dwarf spheroidal galaxies, their mass should be greater than $\sim 300-400eV$ which is well above the cosmologically allowed mass range. Finally the recent Mainz and Troisk experiments from tritium beta decay, combined with neutrino oscillation experiments give an upper limit for the neutrino mass of around 2.2-2.5eV [23, 24, 25]. Massive neutrinos are therefore ruled out as a dark matter candidate capable of solving the missing mass problem.

Cold dark matter (CDM), is composed of very massive slowly moving and weakly interacting particles. A plethora of such particles generically arises in particle physics models beyond the standard model quite naturally. This subject [26, 27] has been

studied in great depth and has been shown to agree with observations to a very good degree. The prospects of discovering a CDM particle are high but so far there has not been a well accepted, firm detection. Observationally problems persist on small scales. As simulations show, it is quite problematic to create galaxies with the right halo profile. It also seems hard to explain the slope and normalization of the Tully-Fisher relation, as well as the very small scatter around it. Moreover, CDM falls short to account for a key observation: that the expansion of the Universe is accelerating [9, 10]. Cosmic acceleration therefore seems to call for a new substance, collectively called the Dark Energy, which contributes a "missing energy".

The discovery that the expansion of the Universe is accelerating, gives a new twist to the whole problem, since a particle like CDM cannot cause such a bizarre phenomenon. The Dark Energy, apparently provides for most of the energy density in the Universe today and must have the very peculiar property that it provides negative pressure. Although research has produced many proposals concerning its nature [28], there is as yet no compelling candidate for the Dark Energy. Its mere presence as well as its magnitude is a puzzle for any sensible particle physics model. To quote the Dark Energy Task Force committee report [29] "nothing short of a revolution in our understanding of fundamental physics will be required to achieve a full understanding of the cosmic acceleration". It may well be that this revolution requires nothing short of re-evaluating our theory of gravity.

Given that the law of gravity plays such a fundamental role at every instance where discrepancies have been observed, it is quite possible that the phenomena commonly attributed to dark matter and dark energy are really a different theory of gravity in disguise. This direction of research [30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42], has remained largely unexplored compared to the extensive treatment that Dark Matter (and more recently Dark Energy) has received.

Most research which concerns modifications to gravity as an explanation to dark matter has revolved around Milgrom's Modified Newtonian Dynamics (MOND) proposal [43, 44, 45]. Within the MOND paradigm, an acceleration scale a_0 is introduced. For accelerations smaller than a_0 , Newton's 2nd law is modified such that the gravitational force is proportional to the square of the particle's acceleration. This simple amendment, has had tremendous success in explaining galactic rotation curves [46, 47, 48, 49, 50, 51, 52] and gives a natural explanation for the Tully-Fisher relation [53, 54]. The reader is referred to [55, 56, 57, 58] for reviews of MOND and to [59] for a thorough study of field theoretical formulations of MOND.

To be able to make predictions for cosmological observations, a relativistic theory is needed. There have been a few attempts to create a relativistic theory that would encompass the MOND paradigm. Starting with a reformulation of MOND as a non-relativistic theory stemming from an action [31], Bekenstein and Milgrom considered the first relativistic realization of MOND by using a scalar field [31]. However, that original formulation was immediately found to have problems with superluminal propagation. A second attempt [33] was ruled out by solar system tests. Both of those theories were based on the existence of two metrics, related by a conformal transformation. It was realized that the simple conformal transformation should be changed. The reason was that any theory based on a conformal transformation would not be able to explain the extreme bending of light observed by massive objects from which very large mass-to-light ratios were being inferred, without additional dark matter. As investigated by Bekenstein, one can change the conformal relation to a disformal one [60] by including an additive tensor in the transformation, not related

to the two metrics, for example build out of the gradient of a scalar field. However, it was soon recognised that any generalized scalar-tensor gravitation theory, even with a disformal relation between the two metrics in the theory, would produce less bending of light than GR and thus could not be used as a basis for relativistic MOND [61].

Sanders finally solved the lensing problem by introducing a unit-timelike vector field into the disformal transformation, in addition to the scalar field [35]. The vector field in Sanders's theory is however non-dynamical, which contradicts the spirit of general covariance. Bekenstein generalized the Sanders theory by making the vector field dynamical, after postulating that its field equations stem from a Maxwellian-type action [37]. The resulting theory was called Tensor-Vector-Scalar (TeVeS) gravitational theory (see [62, 63, 64, 65] for other reviews) and was shown in the same paper to provide for a MOND and a Newtonian limit in the weak field non-relativistic regime, to be devoid of acausal propagation of perturbations, to be in agreement with solar system tests and to produce the right bending of light. This theory is the subject of this review.

This review is organized as follows. In the section 2 I present the theory in its generality by discussing its dynamical elements and giving the actions and field equations. I also give a concise derivation of the quasistatic non-relativistic limit which leads to MOND. In section 3 I discuss the studies of TeVeS regarding homogeneous and isotropic cosmology. In section 4 I move on the discussion of Linear Cosmological Perturbation Theory and focus on a striking TeVeS prediction: that the vector field introduced in the theory to explain gravitational lensing, is also driving cosmological structure formation. In section 5 I present various non-cosmological studies of TeVeS theory, namely spherically symmetric solutions such as Black Holes and neutron stars, stability of spherically symmetric perturbations and gravitational collapse, Parameterized Post-Newtonian constraints, galactic rotation curves, gravitational lensing, superluminality and the time-travel of gravitational waves. Having described the theory and its predictions, in section 6 I give an overview of how the theory was constructed, and give the motivation for its various elements. In section 7 I discuss various other variants of the theory and spin-offs. I conclude the review with an outlook, open questions and future prospects in section 8.

2. Fundamentals of TeVeS

2.1. What are the dynamical agents?

In General Relativity, the spacetime metric g_{ab} is the sole dynamical agent of gravity. Scalar-Tensor theories extend this by adding a scalar field as a dynamical field mediating a spin-0 gravitational interaction. TeVeS also has extra degrees of freedom, but in addition to a scalar field ϕ , there exists a (dual) vector field A_a which also participates into the gravitational sector. Like GR, it obeys the Einstein equivalence principle, but unlike GR it violates the strong equivalence principle.

The original and common way to specify TeVeS theory is to write the action in a mixed frame. That is, we write the action in the "Bekenstein frame" for the gravitational fields, and in the physical frame, for the matter fields. In this way we ensure that the Einstein equivalence principle is obeyed. The three gravitational fields are the metric \tilde{g}_{ab} (with connection $\tilde{\nabla}_a$) that we refer to as the Bekenstein metric, the Sanders vector field A_a and the scalar field ϕ . To ensure that the Einstein equivalence principle is obeyed, we write the action for all matter fields, using a single physical

metric g_{ab} (with connection ∇_a) that we call the "universally coupled metric" ‡. The universally coupled metric is algebraically defined via a disformal relation [60] as

$$g_{ab} = e^{-2\phi} \tilde{g}_{ab} - 2\sinh(2\phi)A_a A_b. \tag{1}$$

The vector field is further enforced to be unit-timelike with respect to the Bekenstein metric, i.e.

$$\tilde{g}^{ab}A_aA_b = -1\tag{2}$$

The unit time-like constraint is a phenomenological requirement for the theory to give the right bending of light (see section 6). Using the unit time-like constraint (2) it is easy to show that the disformal transformation for the inverse metric is

$$g^{ab} = e^{2\phi}\tilde{g}^{ab} + 2\sinh(2\phi)A^aA^b \tag{3}$$

where $A^a = \tilde{g}^{ab} A_b$.

2.2. Action for TeVeS

The theory is based on an action S, which is split as $S = S_{\tilde{g}} + S_A + S_{\phi} + S_m$, where $S_{\tilde{g}}, S_A, S_{\phi}$ and S_m are the actions for \tilde{g}_{ab} , vector field A_a , scalar field ϕ and matter respectively.

As already discussed, the action for \tilde{g}_{ab} , A_a and ϕ is written using only the Bekenstein metric \tilde{g}_{ab} and not g_{ab} , and is such that $S_{\tilde{q}}$ is of Einstein-Hilbert form

$$S_{\tilde{g}} = \frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} \,\tilde{R},\tag{4}$$

where \tilde{g} and \tilde{R} are the determinant and scalar curvature of $\tilde{g}_{\mu\nu}$ respectively and G is the bare gravitational constant. The relation between G and the measured Newton's constant G_N will be elaborated in a subsection 2.4.

The action for the vector field A_a is given by

$$S_A = -\frac{1}{32\pi G} \int d^4x \, \sqrt{-\tilde{g}} \, \left[K F^{ab} F_{ab} - 2\lambda (A_a A^a + 1) \right], \tag{5}$$

where $F_{ab} = \nabla_a A_b - \nabla_b A_a$ leads to a Maxwellian kinetic term and λ is a Lagrange multiplier ensuring the unit-timelike constraint on A_a and K is a dimensionless constant. Indices on F_{ab} are moved using the Bekenstein metric, i.e. $F_b^a = \tilde{g}^{ac} F_{cb}$.

The action for the scalar field ϕ is given by

$$S_{\phi} = -\frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} \left[\mu \, \hat{g}^{ab} \tilde{\nabla}_a \phi \tilde{\nabla}_b \phi + V(\mu) \right] \tag{6}$$

where μ is a non-dynamical dimensionless scalar field, \hat{g}^{ab} is a new metric defined by

$$\hat{g}^{ab} = \tilde{g}^{ab} - A^a A^b, \tag{7}$$

and $V(\mu)$ is an arbitrary function which typically depends on a scale ℓ_B . Not all choices of $V(\mu)$ would give the correct Newtonian or MOND limits in a quasistatic situation. The allowed choices are presented in subsection 2.4. The metric \hat{g}^{ab} is used in the scalar field action, rather than \tilde{g}^{ab} to avoid superluminal propagation of perturbations (see section 6). It is possible to write the TeVeS action using \hat{q}^{ab} ,

‡ Some work on TeVeS, including the original articles by Sanders [35] and Bekenstein [37], refers to the Bekenstein frame metric as the geometric metric and is denoted as g_{ab} , while the universally coupled metric is referred to as the physical metric and is denoted as \tilde{g}_{ab} . Since it is more common to denote the metric which universally couples to matter as g_{ab} , in this review we interchange the tilde.

with the consequence of having more general vector field kinetic terms (see appendix Appendix B.3).

The matter is coupled only to the universally coupled metric g_{ab} and thus its action is of the form

$$S_m[g,\chi^A,\nabla\chi^A] = \int d^4x \sqrt{-g} L[g,\chi^A,\nabla\chi^A]$$
 (8)

for some generic collection of matter fields χ^A . This defines the matter stress-energy tensor T_{ab} through $\delta S_m = -\frac{1}{2} \int d^4x \sqrt{-g} T_{ab} \delta g^{ab}$.

2.3. The field equations

The field equations are easily found using the variational principle. We get two constraint equations, namely the unit-timelike constraint (2) and the μ -constraint

$$\hat{g}^{ab}\nabla_a\phi\nabla_b\phi = -\frac{dV}{d\mu}. (9)$$

which is used to find μ as a function of $\nabla_a \phi$.

The field equations for \tilde{g}_{ab} are given by

$$\tilde{G}_{ab} = 8\pi G \left[T_{ab} + 2(1 - e^{-4\phi}) A^c T_{c(a} A_{b)} \right]
+ \mu \left[\tilde{\nabla}_a \phi \tilde{\nabla}_b \phi - 2 A^c \tilde{\nabla}_c \phi A_{(a} \tilde{\nabla}_{b)} \phi \right] + \frac{1}{2} \left(\mu V' - V \right) \tilde{g}_{ab}
+ K \left[F^c_{\ a} F_{cb} - \frac{1}{4} F^{cd} F_{cd} \tilde{g}_{ab} \right] - \lambda A_a A_b$$
(10)

where \tilde{G}_{ab} is the Einstein tensor of \tilde{g}_{ab} .

The field equations for the vector field A_a are

$$K\tilde{\nabla}_c F^c_{\ a} = -\lambda A_a - \mu A^b \tilde{\nabla}_b \phi \tilde{\nabla}_a \phi + 8\pi G (1 - e^{-4\phi}) A^b T_{ba}, \tag{11}$$

and the field equation for the scalar field ϕ is

$$\tilde{\nabla}_a \left[\mu \hat{g}^{ab} \tilde{\nabla}_b \phi \right] = 8\pi G e^{-2\phi} \left[g^{ab} + 2e^{-2\phi} A^a A^b \right] T_{ab}. \tag{12}$$

The Lagrange multiplier can be solved for by contracting (11) with A^a .

2.4. Recovery of MOND in the quasistatic limit

2.4.1. The quasistatic limit To recover the quasistatic limit, we impose the same assumptions as in the Parameterized Post-Newtonian (PPN) formalism [66], i.e. that the gravitational field is a small fluctuation about a background Minkowski spacetime and that the matter fields can be represented by an effective perfect fluid with density ρ , internal energy Π_E , pressure P and 3-velocity \vec{v} . We then expand all fields in a perturbative series of successive orders in $v = |\vec{v}|$. As in the PPN formalism, we assume that $\frac{\partial}{\partial t} \sim O(v)$, $\Phi_P \sim \rho \sim \Pi_E \sim O(v^2)$ and $P \sim O(v^4)$, where Φ_P is the Poisson potential constructed out of the matter density (which in the case of TeVeS is purely baryonic) and which by definition obeys the Poisson equation $\vec{\nabla}^2 \Phi_P = 4\pi G_N \rho$ with Newton constant G_N . Finally note that to recover the quasistatic limit we only need to keep terms up to $O(v^2)$, and we thus ignore P, and $\rho\Pi_E$ which are higher order. We may then take the stress-energy tensor of matter to be that of a pressureless fluid, i.e. $T_{ab} = \rho u_a u_b$, where u^a is the 4-velocity of the fluid normalized such that $q_{ab}u^au^b = -1$.

We now specify the remaining variables to $O(v^2)$. We have the universally coupled metric as

$$ds^{2} = -(1 + 2\Phi_{N})dt^{2} + (1 - 2\gamma_{PPN}\Phi_{N})\delta_{ij}dx^{i}dx^{j}$$
(13)

where γ_{PPN} is a constant (one of the ten PPN parameters). The fluid velocity is perturbed as $u^{\mu} = (1 + \frac{1}{2}v^2 - \Phi_N, \vec{v})$ and $u_{\mu} = (-1 - \frac{1}{2}v^2 - \Phi_N, \vec{v})$. The geodesic equation for a point particle to $O(v^2)$, gives the non-relativistic acceleration \vec{a} in terms of the potential Φ_N as

$$\vec{a} = \dot{\vec{v}} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla}\Phi_N \tag{14}$$

which is Newton's 2nd law of motion. We thus identify Φ_N with the Newtonian potential.

We turn now to TeVeS for which we have the scalar field as $\phi = \phi_c + \varphi$ and let $\varphi \sim O(v^2)$. The Bekenstein metric is then perturbed as

$$d\tilde{s}^{2} = -e^{-2\phi_{c}}(1 + 2\tilde{\Phi})dt^{2} + e^{2\phi_{c}}(\delta_{ij} + \tilde{h}_{ij})dx^{i}dx^{j}$$
(15)

with $\tilde{\Phi} \sim \tilde{h}_{ij} \sim O(v^2)$. The vector field has components $A_{\mu} = e^{-\phi_c} \left(-1 - \tilde{\Phi}, \vec{0} \right)$ and $A^{\mu} = e^{\phi_c} \left(1 - \tilde{\Phi}, \vec{0} \right)$ where we have assumed that $A_i \sim O(v^3)$. This last assumption requires some further explanation. If we compare A_a with u_a it would seem natural that we should assume that $A_i \sim O(v)$, however, this would also mean that either or both of g_{0i} and \tilde{g}_{0i} should also be of O(v). It turns out, however, that we can always set both $A_i \sim O(v)$ and $\tilde{g}_{0i} \sim O(v)$ (and therefore also g_{0i}) to zero simultaneously by performing a gauge transformation upto a curl §. Therefore we set $A_i \sim \tilde{g}_{0i} \sim g_{0i} \sim O(v^3)$. As the field equations also contain the functions f(X) and $\mu = \frac{df}{dX}$ (see appendix Appendix B.1) or equivalently μ , $V(\mu)$ and $\frac{dV}{d\mu}$, we need to impose the order in v for these as well. It is clear from (B.2) and (9) that $X \sim \frac{dV}{d\mu} \sim O(v^4)$. Furthermore, in the Newtonian limit (see further below) we have that $f(X) \to X$, therefore $f \sim V \sim O(v^4)$. Finally on dimensional grounds we impose $\mu = \frac{df}{dX} \sim O(0)$.

We now proceed to find the quasistatic limit. First we find the (ij)-Einstein equation to $O(v^2)$ which gives $\tilde{G}_{ij} = 0$. By using the disformal transformation to get $\tilde{h}_{ij} = 2[\varphi - \gamma_{PPN}\Phi_N]\delta_{ij}$ we find that $\gamma_{PPN} = 1$. Then we use the vector field equations

§ Consider first the case of GR, and assume that $g_{0i} \sim O(v)$ which leads to the field equation

$$\delta^{kj}\vec{\nabla}_i\vec{\nabla}_k g_{0j} - \vec{\nabla}^2 g_{0i} = 0 \tag{16}$$

from the 0i component of the Einstein equations. Now g_{ab} is not unique but it transforms under gauge transformations generated by a vector field ξ^a . We can choose $\dot{\xi}_0=0$ and $\vec{\nabla}_i\xi_j+\vec{\nabla}_j\xi_i=0$, a choice which leaves g_{00} and g_{ij} invariant but transforms g_{0i} to $g_{0i}+\dot{\xi}_i+\vec{\nabla}_i\xi_0$. Thus if we set g_{0i} to zero by choosing ξ_a such that $g_{0i}=-\dot{\xi}_i-\vec{\nabla}_i\xi_0$ we find that this choice also solves the Einstein equation (16), meaning that $g_{0i}\sim O(v)$ is pure gauge. Therefore we must have that $g_{0i}\sim O(v^3)$. A similar situation arises in TeVeS. If we assume that $A_i\sim O(v)$ we get the field equation

$$\delta^{kj}\vec{\nabla}_i\vec{\nabla}_k A_j - \vec{\nabla}^2 A_i = 0 \tag{17}$$

However, under the special gauge transformation above we have that A_i transforms to $A_i - \vec{\nabla}_i \xi_0$. Furthermore, if $\tilde{g}_{0i} \sim O(v)$ as well, we find that it obeys a field equation like GR (16). It is ease to show that we can simultaneously set both the scalar mode in A_i and \tilde{g}_{0i} to zero by the above gauge transformation, i.e. by choosing $A_i = \vec{\nabla}_i \xi_0$ and $g_{0i} = -\dot{\xi}_i$. This leaves a gauge-invariant purely vector mode, i.e. $\vec{\nabla}_i A_i = 0$ which must therefore be given in terms of a curl as $\vec{A} = \vec{\nabla} \times \vec{H}$. This curl would vanish in spherically symmetric situations and therefore does not have anything to do with the Newtonian or MOND limits. We therefore ignore it and set $A_i \sim \tilde{g}_{0i} \sim g_{0i} \sim O(v^3)$.

to solve for the Lagrange multiplier and insert it to the (00)-Einstein equation to $O(v^2)$ to get

$$\vec{\nabla}^2 \tilde{\Phi} = \frac{8\pi G}{2 - K} \rho. \tag{18}$$

Finally we use the scalar field equation which to $O(v^2)$ gives

$$\vec{\nabla} \cdot \left[\mu \vec{\nabla} \varphi \right] = 8\pi G \rho, \tag{19}$$

while the Newtonian potential, related to the acceleration of particles (14), is given via the disformal transformation as $\Phi_N = \tilde{\Phi} + \varphi$. The above system of equations, (18) and (19) can be solved for any quasistatic situation, regardless of the boundary conditions or the symmetry of the system in question provided a function $\mu(|\vec{\nabla}\varphi|)$ is supplied.

2.4.2. Recovery of Newtonian and MOND limits So far the system of equations (18) and (19) is more general than MOND. Indeed it need not even have any relation to MOND (for example by chosing f = X). However, by considering the two limiting cases of Newtonian gravity for large $|\Phi_N|$ and MOND for small $|\Phi_N|$ we can impose further constaints on the form of f(X). The starting point is to combine (18) and (19) to find a relation between Φ_N and φ . This is readily given up to a curl of a vector \vec{S} as $(2-K)\vec{\nabla}\Phi_N = [2-K+\mu]\vec{\nabla}\varphi + \vec{\nabla}\times\vec{S}$. Thus we arrive at the "AQUAL equation" (see section 6) for Φ_N , namely

$$\vec{\nabla} \cdot \left[\mu_m \vec{\nabla} \Phi_N \right] = 4\pi G_N \rho \tag{20}$$

where μ_m is given by

$$\mu_m = \frac{G_N}{2G} \frac{\mu}{1 + \frac{\mu}{2 - K}} \tag{21}$$

The ratio G_N/G is not free but is found by taking the limit $\mu_m \to 1$, i.e. the Newtonian limit. Consistency then requires that $\mu \to \mu_0$ which is a constant \parallel contained in the function f (or V) and we get the relation [67, 68]

$$\frac{G_N}{G} = \frac{2}{\mu_0} + \frac{2}{2 - K}. (22)$$

The MOND limit is now clearly recovered as $\mu_m \to \frac{|\vec{\nabla}\Phi_N|}{a_0}$. However, due to the presence of the curl, we cannot easily relate Φ_N to φ in general, unless we impose the additional assumption of spherical symmetry for which the curl vanishes. In this case of spherical symmetry, the MOND limit gives

$$\mu \to \frac{2G}{G_N} \frac{|\vec{\nabla}\varphi|}{a_0} = \frac{2G}{G_N} \frac{1}{\ell_B a_0} e^{\phi_c} \sqrt{X}$$
 (23)

where to remind the reader $X = \ell_B^2 \hat{g}^{ab} \nabla_a \phi \nabla_b \phi$. Since $\mu = \frac{df}{dX}$ we may integrate the above equation to find the function f(X) which in the MOND limit should be

$$f \to \frac{2}{3\ell_B a_0} \frac{1}{\frac{1}{\mu_0} + \frac{1}{2-K}} e^{\phi_c} X^{3/2}$$
 (24)

|| The constant μ_0 is related to the constant k intoduced by Bekenstein as $\mu_0 = \frac{8\pi}{k}$.

where the integration constant has been set to zero, as it can always be absorbed into a cosmological constant for the metric \tilde{g}_{ab} . Since both X and f are dimensionless we may define a new constant β_0 such that a_0 is a derived quantity given by

$$a_0 = \frac{2}{3\beta_0 \ell_B} \frac{1}{\frac{1}{\mu_0} + \frac{1}{2-K}} e^{\phi_c} \tag{25}$$

and the function has the MOND limit $f \to \beta_0 X^{3/2}$. Since in the Newtonian limit we also have $f \to \mu_0 X$, there are at least three constants that can appear in f(X), namely μ_0 , β_0 and ℓ_B .

2.4.3. Remark: In terms of the function $\frac{dV}{d\mu}$ the MOND limiting case implies that $\frac{dV}{d\mu} \to -\frac{4}{9\beta_0^2\ell_B^2}\mu^2$ as $\mu \to 0$ while it must diverge as $\mu \to \mu_0$ in the Newtonian limit. This second limit could be imposed if $\frac{dV}{d\mu} \to (\mu_0 - \mu)^{-m}$ for some power m. Bekenstein choses this to be m=1 although other choices are equally valid, even functions that have essential singularities.

2.4.4. Remark: It is clear from (25) that a_0 depends on the cosmological boundary condition ϕ_c which can defer for each system depending on when the system in question was formed. It could thus be considered as a slowly varying function of time. This possibility has been investigated by Bekenstein and Sagi [69] and by Limbach, Psaltis and Feryal [70].

 $2.4.5.\ Remark$: The two limiting cases for f(X) are somewhat strange. In particular we require that $f(X) \to X$ for $X \gg 1$ to recover the Newtonian limit, and that $f(X) \to X^{3/2}$ for $X \ll 1$, in other words a higher power, to recover the MOND limit. This signifies that in these kind of formulation of relativistic MOND, i.e. in terms of a scalar field, the function f(X) should be non-analytic. It further signifies that f(X) can be expanded in positive powers of \sqrt{X} for small X and in positive powers of $\frac{1}{X}$ for large X but the two expansions cannot be connected. In otherwords it is impossible to perturbatively connect the Newtonian with the MOND regime via a perturbation series in $|\vec{\nabla}\varphi|$.

2.4.6. Remark: The Bekenstein free function in [37] is given in the notation given in this review as

$$\frac{dV}{d\mu} = -\frac{3}{32\pi\ell_B^2 \mu_0^2} \frac{\mu^2 (\mu - 2\mu_0)^2}{\mu_0 - \mu}$$
 (26)

which means that $\beta_0 = \frac{4}{3} \sqrt{\frac{2\pi\mu_0}{3}}$ and thus

$$a_0 = \frac{\sqrt{3}}{2\sqrt{2\pi\mu_0}\ell_B} \frac{1}{\frac{1}{\mu_0} + \frac{1}{2-K}} e^{\phi_c}$$
 (27)

This is in agreement with [69] (the authors of [67] have erroneously inverted a fraction in their definition of a_0).

2.4.7. Remark: One can find constraints on the form of f(X) by considering the first order departures from the Newtonian limit [37]. Based on the results found by Bekenstein [37] we can assume that $\mu = \mu_0 + \mu_1 \epsilon + \frac{1}{2} \mu_2 \epsilon^2 + \ldots$ where $\epsilon = 1/(\ell_B |\vec{\nabla}\varphi|)$. The series coefficients μ_i depend on the choice of the function f and for the case of Bekenstein's function in [37] are $\mu_1 = 0$ and $\mu_2 = -\frac{\mu_0^4 a_0^2 \ell_B^2}{32}$. These coefficients can in principle be used to put constraints on the parameters of the free function. Bekenstein notes that for $\mu_0 > 830$ the deviation of $\nabla \Phi_N$ from the Newtonian value at the orbit of the Earth is $< 5.3 \times 10^{-9}$. Further out, to the orbits of the giant planets, it may lead to the deviations recently named the "Pioneer anomaly" [71].

The use of solar system constraints however should be reworked as the solar system is actually in the external field of the galaxy. It is well known that external field effects are important in MOND and the same should hold for TeVeS.

2.5. Where TeVeS can be different than MOND

It is evident from the discussion of the quasistatic limit that exact MOND behaviour is recovered for exact spherical symmetry. For asymmetric situations, the presence of the curl forbids the formulation of the system of equations (18) and (19) in terms of a single AQUAL equation. One should therefore (at least in principle) solve (18) and (19) on its own, after imposing the Newtonian and MOND limit on the form of f(X) and thus $\mu(X)$.

As Bekenstein argues [37], however, the curl $\vec{\nabla} \times \vec{S}$ can in most situations be neglected, so that an effective MOND law is recovered even in asymmetric systems. Thus the curl should be important in the inner regions or near-exteriors of galaxies but not far away from them. As Bekenstein points out [37] if a system is asymmetric but very dense so that the Newtonian regime applies everywhere, it is quite safe to neglect the curl. Note also that in principle it is also possible to have an O(v) curl coming from the vector field, i.e. $\vec{A} = \vec{\nabla} \times \vec{H} \sim O(v)$. If this is the case, then the $O(v^2)$ equations would have terms involving $|\vec{A}|^2$. This would also give corrections to the MOND (and also to the Newtonian) limit for aspherical systems and since this curl is not sourced by matter, Bekenstein's argument need not apply. Its implications should therefore be investigated further.

We also note that the recovery of MOND is done only in the quasistatic limit. In all situations where this limit does not apply (such as cosmology), we should not expect to get any relation to MOND, at least not in TeVeS theory. Finally, departures from MOND behaviour are to be expected in cases where one must have the equations to $O(v^3)$ or $O(v^4)$, or where the cosmological background should be included as a time-dependent background (which may change the order of some terms). This possibility may include galaxy clusters and deserves further inverstigation. Could the vector field which plays a role for driving large scale structure formation (see below) reconcile the problems of MOND with clusters?

3. Homogeneous and Isotropic cosmology

3.1. FLRW equations

Solutions to the TeVeS equations for a homogeneous and isotropic spacetime described by a Friedmann-Lemaitre-Robertson-Walker (FLRW) metric have been extensively studied [37, 72, 73, 74, 75, 76, 67, 77, 78, 79].

For an FLRW spacetime, the universally coupled metric takes the conventional synchronous form

$$ds^2 = -dt^2 + a^2 \delta_{ij} dx^i dx^j \tag{28}$$

for physical scale factor a and where I have assumed for simplicity that the spatial hypersurfaces are flat (see [76] for the curved case). The Bekenstein metric also has a similar form, i.e. it is given as

$$d\tilde{s}^2 = -d\tilde{t}^2 + b^2 \delta_{ij} dx^i dx^j \tag{29}$$

for a second scale factor b. The disformal transformation relates the two scale factors as $a = be^{-\phi}$ while the two coordinate times t and \tilde{t} are related by $dt = e^{\phi}d\tilde{t}$. The physical Hubble parameter is defined as usual as

$$H = \frac{d\ln a}{dt} = \frac{\dot{a}}{a} \tag{30}$$

while the Bekenstein frame Hubble parameter is $\tilde{H}=\frac{d\ln b}{d\tilde{t}}=\frac{b'}{b}=e^{\phi}H+\phi'$. Cosmological evolution is governed by the Friedmann equation

$$3\tilde{H}^2 = 8\pi G e^{-2\phi} \left(\rho_\phi + \rho\right) \tag{31}$$

where ρ is the physical matter density which obeys the energy conservation equation with respect to the universally coupled metric and where the scalar field energy density is

$$\rho_{\phi} = \frac{e^{2\phi}}{16\pi G} \left(\mu \frac{dV}{d\mu} + V \right) \tag{32}$$

Similarly one can define a scalar field pressure as

$$P_{\phi} = \frac{e^{2\phi}}{16\pi G} \left(\mu \frac{dV}{d\mu} - V \right) \tag{33}$$

The scalar field evolves according to the two differential equations

$$\phi' = -\frac{1}{2\mu}\Gamma\tag{34}$$

and

$$\Gamma' + 3\tilde{H}\Gamma = 8\pi G e^{-2\bar{\phi}}(\rho + 3P) \tag{35}$$

while μ is found by inverting ${\phi'}^2 = \frac{1}{2} \frac{dV}{d\mu}$ It is important to note that the vector field must point to the time direction and its components are given by $A_{\mu} = (\sqrt{\tilde{g}_{00}}, \vec{0})$. Therefore it does not contain any independent dynamical information and it does not explicitly contribute to the energy density. Its only effect is on the disformal transformation which relates the Bekensteinframe Friedmann equation (31) with the physical Friedmann equation. This is true also in cases where the vector field action is generalized and where the only effect is a constant rescaling of the left-hand-side of the Bekenstein-frame Friedmann equation (31) as discussed by Skordis in [78].

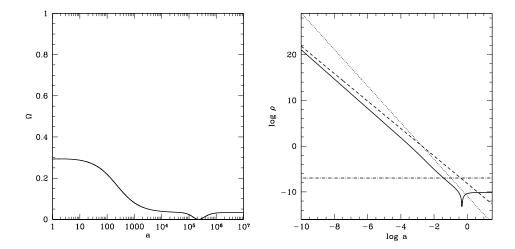


Figure 1. LEFT: The cosmological evolution of the relative density of the scalar field Ω_{ϕ} with $\mu_0=5$ in a universe with radiation, baryons and a cosmologicical constant. The energy densities of the matter components have been adjusted to unrealistic values to show the tracking behavior of the scalar field clearly. RIGHT: The cosmological evolution of the energy densitiy ρ with the scale factor a for baryons (dashed), radiation (dotted), cosmological constant (dot-dash) and scalar field ϕ (solid). Here $\mu_0=250$ and the energy densities of the matter components correspond to realistic values typical in our universe.

3.2. The Bekenstein function

In the original TeVeS paper [37] Bekenstein studied the cosmological evolution of an FLRW universe in TeVeS assuming that the free function is given by (26). He showed that the scalar field contribution to the Friedmann equation is very small, and that ϕ evolves very little from the early universe until today. He noted that with this choice of function, a cosmological constant term has to be added in order to have an accelerated expansion today as required by the SN1a data.

Many cosmological studies, for example [72, 73, 75, 78], have used this Bekenstein function. In particular Hao and Akhoury [72] noted that the integration constant obtained by integrating (26) can be used to get an accelerated expansion and noted that TeVeS has the potential to act as dark energy. However, such an integration constant cannot be distinguished from a bare cosmological constant term in the Bekenstein frame. Thus it is dubious whether this can be interpreted as dark energy arising from TeVeS. Nevertheless, it would not be a surprizing result if some other TeVeS functions could in fact provide for dark energy, because of the close resemblance of the scalar field action in TeVeS and k-essence [80, 81]. Indeed Zhao [77] has investigated this issue further (see below).

Exact analytical and numerical solutions with the Bekenstein free function (26) have been found by Skordis et al. [73] and by Dodelson and Liguori [75]. It turns out that not only, as Bekenstein noted, the scalar field is subdominant, but also its energy density tracks the matter fluid energy density. This kind of tracker behaviour has been found before in scalar field models of dark energy [82, 83, 84, 85, 86, 87, 88, 89] called tracking quintessence. The ratio of the energy density of the scalar field to that of ordinary matter is approximately constant (see left panel of Figure 1), so that the

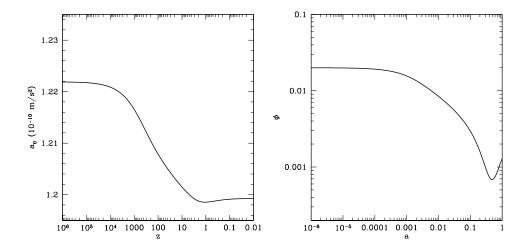


Figure 2. Left: The time evolution of the Mondian acceleration a_0 with redshift, for the Bekenstein function in a typical background cosmological model. Right: The time evolution of ϕ for the Bekenstein function with the scale factor for the same model.

scalar field exactly tracks the matter dynamics. In the case of TeVeS, one gets that

$$\Omega_{\phi} = \frac{(1+3w)^2}{6(1-w)^2\mu_0} \tag{36}$$

where w is the equation of state of the background fluid ¶. Since μ_0 is required to be very large, the scalar field relative energy density is always small, with values typically smaller than $\Omega_{\phi} \sim 10^{-3}$ in a realistic situation. As investigated by Skordis [78] the tracker solutions are also present (for this choice of function) in versions of TeVeS with more general vector field actions.

In realistic situations, the radiation era tracker is almost never realized, as has been noted by Dodelson and Liguori. Rather, during the radiation era, the scalar field energy density is subdominant but slowly growing (see right panel of Figure 1) and the scalar field is given by $\phi \propto a^{4/5}$. Upon entering the matter era it settles into the tracker solution. This transient solution in the radiation era has been generalized by Skordis [78] to an arbitrary initial condition for ϕ , a more general free function (see below) and a general vector field action. It should be stressed that the solution in the radiation era is important for setting up initial conditions for inhomogeneous perturbations about the FLRW solutions, relevant for studying the physics of the CMB radiation and Large Scale Structure (LSS).

Since a_0 for a quasistatic system depends on the cosmological value of the scalar field at the time the system broke off from the expansion and collapsed to a bound structure, it is possible that different systems would exhibit different values of a_0 depending when they formed. Figure 2 displays the evolution of a_0 and the scalar field ϕ for realistic cosmological model. The impact of evolving a_0 on observations has been investigated by [69, 70].

Finally, the sign of ϕ changes between the matter and cosmological constant eras. In doing so, the energy density of the scalar field goes momentarily through zero, since

¶ This excludes the case of a stiff fluid with w=1.

it is purely kinetic and vanishes for zero $\dot{\phi}$ [73].

3.3. Generalizing the Bekenstein function

Bourliot et al [67] studied more general free functions which have the Bekenstein function as a special case. In particular they introduced two new parameters, a constant μ_a and a power index n such that the free function is generalized to

$$\frac{dV^{(n)}}{d\mu} = -\frac{3}{32\pi\ell_B^2\mu_0^2} \frac{\mu^2(\mu - \mu_a\mu_0)^n}{\mu_0 - \mu}$$
(37)

This function $^+$ retains the property of having a Newtonian limit as $\mu \to \mu_0$ and a MOND limit as $\mu \to 0$. Furthermore, as remarked in [67], more general functions can be built by considering the sum of the above prototypical function with arbitrary coefficients, i.e. as $\frac{dV}{d\mu} = \sum_n c_n \frac{dV^{(n)}}{d\mu}$. The cosmological evolution depends on the power index n and can be categorized as follows.

3.3.1. Case $n \geq 1$ Clearly $\frac{dV}{d\mu}(\mu_a\mu_0) = 0$ and at that point $\dot{\phi} \to 0$. Now suppose that the integration constant is chosen such that $V(\mu_a\mu_0) = 0$ as well. Then, just like the case of the Bekenstein function (which is included in this sub-case as n=2 and $\mu_a=2$), one gets tracker solutions. The function μ is driven to $\mu=\mu_a\mu_0$ at which point $\dot{\phi}=0$. There are no oscillations around that point, but it is approached slowly so that it is exactly reached only in the infinity future. The scalar field relative density is given by

$$\Omega_{\phi} = \frac{(1+3w)^2}{3\mu_a(1-w)^2\mu_0} \tag{38}$$

independently of the power index n. It should also be pointed out that the evolution of the physical Hubble parameter H would be different than the case of GR even in the tracking phase [67]. For example in the case w=0 we would have $H \propto a^{-n_h}$ where $n_h = \frac{1+3\mu_a\mu_0}{2(\mu_a\mu_0-1)}$.

Furthermore, just like the Bekenstein case, the radiation era tracker is untenable

Furthermore, just like the Bekenstein case, the radiation era tracker is untenable for realistic cosmological evolution for which μ_0 must be large so that Ω_{ϕ} would be very small (\sim < 10^{-3}). As was shown in [78], in this case we once again get a transient solution where the scalar field evolves as $\phi \propto a^{4/(3+n)}$.

In the case that the integration constant is chosen such that $V(\mu_a \mu_0) \neq 0$ then one has an effective cosmological constant present. Thus once again we get tracker solutions until the energy density of the Universe drops to values comparable with this cosmological constant, at which case tracking comes to an end, and the universe enters a de Sitter phase.

3.3.2. Case $-3 \le n \le 0$ The cases n = 0 to n = -2 turn out to be pathological as they lead to singularities in the cosmological evolution [67]. The case n = -3 is well behaved when the matter fluid is a cosmological constant, however, it also is pathological when the matter fluid has a different equation of state than w = -1 [67].

⁺ Note that [67] uses a different normalization for V and their results can be recovered by rescaling the scale ℓ_B in this report as $\ell_B \to \ell_B \sqrt{\frac{3}{2} \mu_0^{n-3}}$.

3.3.3. Case $n \leq -4$ The cases for which $n \leq -4$ are well behaved in the sence that no singularities occur in the cosmological evolution. Contrary to the $n \geq 1$ cases the cosmological evolution drives the function μ to infinity. Moreover these cases do not display the tracker solutions of $n \geq 1$, but rather the evolution of ρ_{ϕ} is such that it evolves more rapidly than the matter density ρ and so quickly becomes subdominant. The General Relativistic Friedmann equation is thus recovered, i.e. $3H^2 = 8\pi G\rho$. We also get that $\tilde{H} = H$ which means that the scalar field is slowly rolling.

The evolution of the scalar field variables Γ , ϕ and μ then depends on the equation of state of the matter fluid. If the background fluid is a cosmological constant, then we get de Sitter solutions for both metrics and we get that $\Gamma = 2H(e^{-3Ht} - 1)$.

For the case of a stiff fluid with equation of state w=1, we get that Γ has power-law solutions in inverse powers of t as $\Gamma=\frac{6}{t}+\frac{\Gamma_0}{t^3}$. A similar situation arises for -1 < w < 1 for which we get $\Gamma=\frac{2(1+3w)}{1-w}H$ and the Hubble parameter evolves as $H=\frac{2}{3(1+w)}\frac{1}{t}$. Notice that the limit $w\to 1$ for the -1 < w < 1 case does not smoothly lead to the w=1 case.

The observational consequences on the CMB and LSS have not been investigated for this case of function, unlike the case of the Bekenstein function.

3.3.4. Mixed cases Mixing different powers of $n \ge 1$ leads once again to tracker solutions. One may have to add an integration constant in order to keep $V(\mu_a \mu_0) = 0$, although for certain combinations of powers n and coefficients c_i it is not necessary.

Mixing n=0 with some other $n \geq 1$ cannot remove the pathological situation associated with the n=0 case. Mixing n=0 with both positive and negative powers could however lead to acceptable cosmological evolution since the effect of the negative power is to drive μ away from the $\mu = \mu_a \mu_0$ point.

In general if we mix an arbitrary number of positive and negative powers we would get tracker solutions provided we could expand the new function in positive definite powers of $\mu - \mu'_a \mu_0$ where μ'_a is some number different from the old μ_a .

3.4. Inflationary/accelerated expansion solutions

Diaz-Rivera, Samushia and Ratra [74] have studied cases where the cosmological TeVeS equations lead to inflationary/accelerated expansion solutions. They first consider the vacuum case, where the matter density ρ vanishes. In that case, they find that one gets de Sitter solutions $b \sim e^{\tilde{H}_0 \tilde{t}}$ where the Bekenstein frame Hubble constant \tilde{H}_0 is given by the free function as $\tilde{H}_0 = \sqrt{\frac{\mu_0^2 V}{6}}$ and where $\frac{dV}{d\mu} = 0$, i.e. the scalar field is constant $\phi = \phi_i$. It is clear that such a solution will always occur (in vacuum) provided the free function satisfies $\frac{dV}{d\mu}(\mu_v) = 0$ and $V(\mu_v) \neq 0$ for some constant μ_v . In that case, the general solution is obviously not de Sitter since both ϕ and μ will be time-varying but will tend to this vacuum solution as $\mu \to \mu_v$. Indeed the $n \geq 1$ case of Bourliot $et\ al\ [67]$ with an integration constant is precisely this kind of situation.

In the non-vacuum case, for which the matter fluid has equation of state $P=w\rho$, they make the ansatz $b^{3(1+w)}=e^{(1+3w)\phi}$ which brings the Friedmann equation into $3\tilde{H}^2=8\pi G\rho_0+\frac{1}{2}(\mu V'+V)$, where ρ_0 is the matter density at a=1. Once again they assume that the free-function-dependent general solution drives μ to a constant μ_v but ϕ is evolving. Thus we must have that $\phi=\phi_1\tilde{t}+\phi_2$, such that $\dot{\phi}=\phi_1$ is a constant. In order for ϕ_1 to be non-zero we must have $\frac{dV}{d\mu}(\mu_v)\neq 0$. However, there is a drawback of this situation. As they point out, consistency with the scalar

field equation requires that w < -1. Furthermore, although this solution is a de Sitter solution in the Bekenstein-frame, it corresponds to a power-law solution for the universally coupled metric. In order for this power-law solution to lead to acceleration, they find that -5/3 < w < -1. Since no known fluids exist in this range of w this solution is of dubious importance.

3.5. Accelerated expansion in TeVeS

The simplest case of accelerated expansion in TeVeS is provided by using a cosmological constant term. This is equivalent to adding an integration constant to $V(\mu)$ [72, 67] and it corresponds to the accelerated expansion considered by Diaz-Rivera, Samushia and Ratra [74] in both the vacuum or non-vacuum cases (above). This kind of solution does not make things any better than a cosmological constant model and all of its associated problems and coincidences. It would be therefore be of importance to find accelerated solutions in TeVeS without such a constant, simply by employing the scalar field (these need not be de Sitter solutions).

Zhao used a simple function $\frac{dV}{d\mu} \propto \mu^2$ to obtain solutions which provide acceleration, and compared his solution with SN1a data [77], finding good agreement. However, it is not clear whether other observables such as the CMB angular power spectrum or observations of Large Scale Structure are compatible with this function. Furthermore, this function is certainly not realistic as it does not have a Newtonian limit (it is always MONDian).

Although no further studies of accelerated expansion in TeVeS have been performed, it is very plausible that certain choices of function will inevitable lead to acceleration. It is easy to see that the scalar field action has the same form as a k-essence/k-inflation [80, 81] action which has been considered as a candidate theory for acceleration. More precisely, the Friedmann-fluid-scalar field system of cosmological equations corresponds to k-essence coupled to matter. It is unknown in general whether this has similar features as the uncoupled k-essence, although Zhao's study indicates that this a promising research direction. Let us also note that disformal transformations can also play a crucial role in theories of acceleration even for canonical scalar field actions as investigated by Koivisto with disformal quintessence [90].

3.6. Realistic FLRW cosmology

In TeVeS, cold dark matter is absent. Therefore in order to get acceptable values for the physical Hubble constant today, we have to supplement the absence of CDM with something else. Possibilities include the scalar field itself, massive neutrinos [73, 79] and a cosmological constant. At the same time, one has to get the right angular diameter distance to recombination [79]. These two requirements can place severe constraints on the allowed free functions.

4. Linear cosmological perturbation theory in TeVeS

4.1. Scalar modes

The full linear cosmological perturbation theory in TeVes has been worked out by Skordis [76] as well as for variants of TeVeS with more general vector field actions [78].

The scalar modes of the linearly perturbed universally coupled metric are given in the conformal Newtonian gauge as

$$ds^{2} = -a^{2}(1+2\Psi)d\tau^{2} + a^{2}(1-2\Phi)\delta_{ij}dx^{i}dx^{j}$$
(39)

where for simplicity we have assumed that the spatial curvature is zero and where τ is the conformal time defined as $dt = ad\tau$. See [76, 78] for the curved cases, as well as the cases of vector and tensor perturbations. The scalar field is perturbed as $\phi = \bar{\phi} + \varphi$ where $\bar{\phi}$ is the FLRW background scalar field and φ is the perturbation. The vector field is perturbed as $A_{\mu} = ae^{-\bar{\phi}}(\Psi - \varphi, \vec{\nabla}_i \alpha)$ such that the unit-timelike constraint is satisfied, which removes the A_0 component as an independent dynamical degree of freedom. Thus there are two additional dynamical degrees of freedom to GR, that is the scalar field perturbation φ and the vector field scalar mode α .

The field equations for the scalar modes can be found in the conformal Newtonian gauge in [73], in general gauges in including the synchronous gauge in [76] and for more general TeVeS actions in [78].

4.2. Initial conditions for the scalar modes

Setting up initial conditions for perturbations in cosmology has traditionally been classified in terms of adiabatic and isocurvature modes. In the Λ CDM model five regular modes have been identified [91], namely the adiabatic growing mode, the Baryon Isocurvature Density mode, the CDM Isocurvature Density mode, the Neutrino Isocurvature Velocity mode.

Generating initial conditions has always been one of the most important issue in cosmology. One theory which generates initial conditions is inflation. Typically single-field inflationary models predict an adiabatic spectrum of fluctuations, however, more general multi-field inflationary models or models with curvatons predict a mixture of sometimes uncorrelated, and sometimes correlated, adiabatic and isocurvature modes. Although generating the different modes is important, the issue of their observability can be dealt with separately, which in turn can place constraints on the theory which generates them. Various multi-parameter studies of initial conditions and their observational impact on the CMB radiation and the LSS have limited the total contribution from isocurvature modes to less than 30% when all cases of arbitrarily correlated modes are allowed [92, 93, 94] and to a few percent in the case when an uncorrelated single mode mixed with the adiabatic mode is allowed [95, 96, 92, 93, 97, 94, 98, 99, 100]. Thus in a Λ CDM cosmology, the dominant contribution must be adiabatic to a large extend.

The exact adiabatic growing mode in TeVeS and generalized variants has been found by Skordis in [78], but only for the case of the generalized Bekenstein function. In general, the correct setup of initial conditions would depend on the free function. If, however, the free function is such that the scalar field contribution to the background expansion, during the radiation era is very small, then the adiabatic mode for other free functions would only marginally differ from the ones found in [78]. In particular, the only effect would be a difference in the initial conditions of φ and therefore it is unlikely that this would make any observational difference.

The only study of observational signatures of TeVeS theory in the CMB radiation and the LSS, namely that of Skordis, Mota, Ferreira and Bæhm [73] have used initial conditions such both φ and α as well as their derivatives are zero initially. While this is not a pure adiabatic initial condition, it turns out that it is close enough to the

adiabatic initial conditions found in [78] so that no observable difference can be seen from any isocurvature contamination.

Studies of isocurvature modes in TeVeS have not been conducted. In the light of the problems that TeVeS has with observations of the CMB radiation [73] it may be important to investigate what the observational effects are that the isocurvature modes would have. For example correlated mixtures of adiabatic and isocurvature modes could lower the Integrated Sachs-Wolfe effect or raise the 3rd peak both of which pose significant problems to TeVeS. Preliminary studies by Mota, Ferreira and Skordis have shown that setting the vector field perturbations large initially can have significant impact on both of these features [101].

In addition to the four regular isocurvature modes that exist in GR as mentioned above, there could in principle exist four other isocurvature modes: two associated with the scalar field and two associated with the vector field. Preliminary studies by Skordis have shown that none of the scalar field isocurvature modes are regular [102]. Conversely, under certain conditions of the vector field parameters one of the vector field isocurvature modes can be regular while the other one is never regular. Thus, it may be possible to have one regular isocurvature mode in the TeVeS sector. The observational consequences of this mode are unknown as well as its generation method from theories such as inflation in the presence of TeVeS fields.

4.3. Linear equations for vector and tensor modes

The full linear cosmological perturbation formalism for vector and tensor modes has been worked out in [76] as well as for variants of TeVeS with more general vector field actions [78]. No studies of observable spectra based on vector or tensor modes have been conducted.

4.4. Large Scale Structure observations

A traditional criticism of MOND-type theories was their lack of a dark matter component and therefore their presumed inability to form large scale structure compatible with current observational data. This incorrect reasoning was based on a General Relativistic universe filled only with baryons. In that case it is well known that, since baryons are coupled to photons before recombination they do not have enough time to grow into structures on their own. Furthermore, their oscillatory behaviour at recombination is preserved and is visible as large oscillation in the observed galaxy power spectrum $P_{gg}(k)$. Finally, on scales smaller than the diffusion damping scale they are exponentially damped due to the Silk-damping effect. CDM solves all of these problems because it does not couple to photons and therefore can start creating potential wells early on, to which the baryons fall into. This is enough to generate the right amount of structure, largely erase the oscillations and overcome the Silk damping.

However, TeVeS is not General Relativity. It contains two additional fields, which change the structure of the equations significantly. The first study of TeVeS predictions for Large Scale Structure observations was conducted by Skordis, Mota, Ferreira and Bæhm [73]. They numerically solved the perturbed TeVeS equations for the case of the Bekenstein function and determined the effect on the matter power spectrum P(k). They found that TeVeS can indeed form large scale structure compatible with observations depending on the choice of TeVeS parameters in the free function. In fact

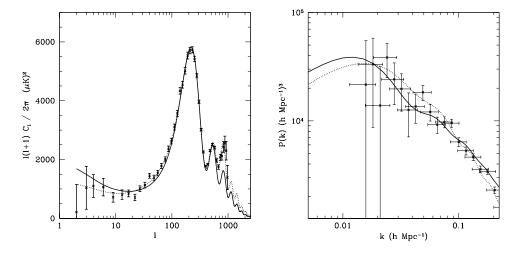


Figure 3. LEFT: The Cosmic Microwave Background angular power spectrum $l(l+1)C_l/(2\pi)$ for TeVeS (solid) and Λ CDM (dotted) with WMAP 5-year data [8]. RIGHT: The matter power spectrum P(k) for TeVeS (solid) and Λ CDM (dotted) plotted with SDSS data.

the form of the matter power spectrum P(k) in TeVeS looks quite similar to that in Λ CDM. Thus TeVeS can produce matter power spectra that cannot be distinguished from Λ CDM. One would have to turn to other observables to distinguish the two models. The power spectra for TeVeS and Λ CDM are plotted on the right panel of Figure 3.

Dodelson and Liguori [75] provided an analytical explanation of the growth of structure seen numerically by [73]. They have found that the growth in TeVeS cannot be due to the scalar field. In fact the scalar field perturbations have Bessel function solutions and are decaying in an oscillatory fashion. Instead, they found that the growth in TeVeS is due to the vector field perturbation.

Let us see how the vector field leads to growth. Using the tracker solutions in the matter era from Bourliot et al [67] we find the behaviour of the background functions a,b and $\bar{\phi}$. These are used into the perturbed field equations, after setting the scalar field perturbations to zero, and we find that in the matter era the vector field scalar mode α obeys the equation

$$\ddot{\alpha} + \frac{b_1}{\tau}\dot{\alpha} + \frac{b_2}{\tau^2}\alpha = S(\Psi, \dot{\Psi}, \theta) \tag{40}$$

in the conformal Newtonian gauge, where

$$b_1 = \frac{4(\mu_0 \mu_a - 1)}{\mu_0 \mu_a + 3} \tag{41}$$

$$b_2 = \frac{2}{(\mu_0 \mu_a + 3)^2} \left[\mu_0^2 \mu_a^2 - \left(5 + \frac{4}{K}\right) \mu_0 \mu_a + 6 \right]. \tag{42}$$

and where S is a source term which does not explicitly depend on α . If we take the simultaneous limit $\mu_0 \to \infty$ and $K \to 0$ for which $\Omega_\phi \to 0$ meaning that the TeVeS contribution is absent, we get $b_1 \to 4$ and $b_2 \to 2$. In this case the two homogeneous solutions to (40) we τ^{-2} and τ^{-1} which are decaying. Dodelson and Liguori show

that the source term $S(\Psi, \dot{\Psi}, \theta)$ is not sufficient to create a growing mode out of the general solution to (40) and therefore in this General Relativistic limit, TeVeS does not provide enough growth for structure formation.

Now lets look at the general case. Dodelson and Liguori assume the ansatz that the homogeneous solutions to (40) are given as τ^n for some power index n. Generalizing their result to the generalized Bekenstein function of Bourliot $et\ al\ [67]$ we get that

$$n = \frac{-3 + \frac{7}{\mu_0 \mu_a} \pm \sqrt{1 + \frac{1}{\mu_0^2 \mu_a^2} - \frac{2}{\mu_0 \mu_a} + \frac{32}{K \mu_0 \mu_a}}}{2(1 + \frac{3}{\mu_0 \mu_a})}$$
(43)

Now since $\Omega_{\phi} \sim 10^{-3}$ we have that $\frac{1}{\mu_0 \mu_a} \sim 3 \times 10^{-3}$ and we may ignore these terms to get

$$n \approx -\frac{3}{2} + \frac{1}{2}\sqrt{1 + \frac{32}{K\mu_0\mu_a}} \tag{44}$$

where we have ignored the negative sign which would give a decaying mode. Thus we can get n > 0 provided

$$K < \sim 0.01 \tag{45}$$

for fixed $\mu_0\mu_a$. Smaller values of $\mu_0\mu_a$ can also raise this threshold. Thus if this condition is met, there exists a growing mode in α . This in turn feeds back into the perturbed Einstein equation and sources the gravitational potential Φ . This is translated into a non-decaying mode in Φ which in turn drives structure formation. This is graphically displayed in the left panel of Figure 4.

It is a striking result that even if the contribution of the TeVeS fields to the background FLRW equations is negligible ($\sim 10^{-3}$ or less), we can still get a growing mode which drives structure formation. This explains analytically the numerical results of Skordis, Mota, Ferreira and Bohm [73].

4.5. Cosmic Microwave Background observations

A General Relativistic universe dominated by baryons cannot fit the most up to date observations of the CMB anisotropies [8]. This is true even if a cosmological constant and/or three massive neutrinos are incorporated into the matter budget so that the first peak of the CMB angular power spectrum is at the right position. Although a model with baryons, massive neutrinos and cosmological constant can give the correct first and second peaks, it gives a third peak which is lower than the second. On the contrary, the measured third peak is almost as high as the second. This is possible if CDM is present.

Once again, TeVeS is not General Relativity and it is premature to claim (as in [103, 104]) that only a theory with CDM can fit CMB observations. In fact as was shown by Bañados, Ferreira and Skordis [105], the Eddington-Born-Infeld (EBI) theory [41, 42] can produce CMB spectra that look like Λ CDM. Although the EBI theory has no relation to TeVeS and has features which are rather different than TeVeS, it is not inconceivable that TeVeS theory or some variant could also give similar results. Skordis, Mota, Ferreira and Bœhm [73] numerically solved the linear Boltzmann equation in the case of TeVeS and calculated the CMB angular power spectrum for TeVeS. By using initial conditions close to adiabatic the spectrum thus found provides very poor fit as compared to the Λ CDM model (see the left panel of

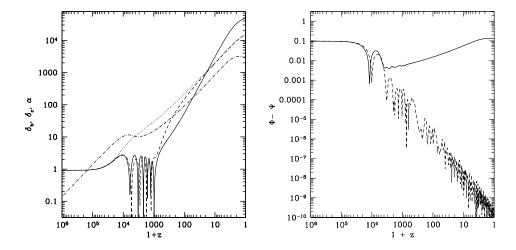


Figure 4. LEFT: The evolution of the baryon density fluctuation in TeVeS (solid) and Λ CDM (dashed) with redshift for a wavenumber $k=10^{-3}Mpc^{-1}$. Notice how in both cases, baryons fluctuate before recombination and grow afterwards. In the case of Λ CDM, the baryons eventually follow the CDM density fluctuation (dotted) which starts growing before recombination. In the case of TeVeS, the baryons grow due to the potential wells formed by the growing scalar mode in the vector field α (dash-dot).

RIGHT: The difference of the two gravitational potentials $\Phi - \Psi$ for a wavenumber $k = 10^{-3} Mpc^{-1}$ plotted against redshift. The solid curve is a TeVeS model which gives the correct matter power spectrum. The dotted curve is a Λ CDM model.

Figure 3). The CMB seems to put TeVeS into trouble, at least for the Bekenstein free function. If this was a complete study then TeVeS would already be ruled out by the CMB data.

It may be that different variants of TeVeS, e.g. with different vector field actions, or different scalar free functions could give better fits, but at the moment this is still an open problem. A different possibility is the use of correlated isocurvature modes, in particular modes from the TeVeS sector. This will undoubtedly give different spectra but once again the question of whether it will lead to acceptable spectra is open.

Needless to say, if none of the aforementioned possibilities works it would quite possibly be the end of TeVeS as a theory capable of explaining the mass and energy discrepancies on galactic, cluster and cosmological scales.

We should note however, that the presence of a fourth sterile neutrino with mass $\sim 11 eV$ will give very good fits to the CMB angular power spectrum, as found by Angus [106]. This is true also in General Relativity and so this result has nothing to do with TeVeS features. However, in General Relativity this neutrino would severely suppress Large Scale Structure, and only in TeVeS there is the possibility to counteract it.

4.6. Sourcing a difference in the two gravitational potentials : $\Phi - \Psi$

The result of Dodelson and Liguori [75] has a further direct consequence. The perturbed TeVeS equations which relate the difference of the two gravitational potentials $\Phi - \Psi$ to the shear of matter, have additional contributions coming from the

perturbed vector field α . This is not due to the existence of the vector field per se but comes from the disformal transformation to which the vector field plays a fundamental role. Indeed in a single metric theory where the vector field action is Maxwellian (just like TeVeS), there is no contribution of the vector field to $\Phi - \Psi$.

Since the vector field is required to grow in order to drive structure formation, it will inevitably lead to a growing $\Phi - \Psi$. This is precisely what we see numerically in the right panel of Figure 4.

If the difference $\Phi - \Psi$, named the gravitational slip, can be measured observationally, it can provide a substantial test of TeVeS that can distinguish TeVeS from Λ CDM. The link between $\Phi - \Psi$ and theories of gravity has been noted before by Lue, Starkman and Scoccimarro [107] and by Bertschinger [108]. Since Dodelson and Liguori uncovered its importance for theories like TeVeS, many authors [109, 110, 111, 112, 113, 114] have investigated various observational techniques to probe the gravitational slip. Theoretical consideration on general tests of modifications to gravity and their effect on observations have been investigated in [115, 116, 117].

4.7. Inflation and TeVeS

In TeVeS, the disformal transformation plays a fundamental role. It may be that this can lead to interesting features in the initial power spectrum and/or the production of isocurvature modes in the TeVeS sector, in an inflationary universe. Disformal inflation has been investigate before by Kaloper [118] but inflation in the context of TeVeS remains an unexplored direction. If isocurvature perturbations are found to significantly contribute to the CMB anisotropies, it would be of particular importance to check whether they can be produced in an inflationary era in TeVeS.

5. Non-cosmological studies of TeVeS theory

It is not the main focus of this review to present non-cosmological studies of TeVeS theory, however, in this section I give a brief outline of the work done along a variety of directions. This includes spherically symmetric systems, such as Black Holes and neutron stars, galactic rotation curves, gravitational lensing, galaxy clusters and galaxy groups, post-Newtonian parameters and theoretical issues such as stability and singularities.

5.1. Spherically symmetric systems

5.1.1. Black holes Static, spherically symmetric vacuum systems were first considered by Bekenstein in the original TeVeS paper [37], in order to calculate some of the PPN parameters for TeVeS. The first spherically symmetric static vacuum solutions in TeVeS, however, were thoroughly studied by Giannios [119]. Under the assumption of strong field limit $\mu = \mu_0$, he found a family of vacuum solutions described in isotropic coordinates by the physical metric

$$ds^{2} = -\left(\frac{r - r_{c}}{r + r_{c}}\right)^{n} dt^{2} + \frac{(r^{2} - r_{c}^{2})^{2}}{r^{4}} \left(\frac{r - r_{c}}{r + r_{c}}\right)^{-n} \left(dr^{2} + r^{2} d\Omega^{2}\right)$$
(46)

where r_c is an integration constant of dimensions of length and the power n is given by

$$n = 2\sqrt{\frac{1 - \frac{1}{2\mu_0} \frac{GM_s}{r_c}}{1 - \frac{K}{2}}} + \frac{2}{\mu_0} \frac{GM_s}{r_c}.$$
 (47)

The mass M_s , called the scalar mass, is determined in the case where the above solution is an exterior solution to a star as

$$M_s = \int d\Omega \int_0^R dr \sqrt{-g} (\rho + 3P) \tag{48}$$

In the above solution, it was assumed that the vector field is aligned with the time direction and is thus fixed by the unit-norm constraint. The scalar field was found to have the profile

$$\phi = \phi_c + \frac{GM_s}{\mu_0 r_c} \ln \frac{r - r_c}{r + r_c} \tag{49}$$

Although the above solution describes an exterior solution, it does not necessarily describe a Black Hole. This is to be expected since there is no Jebsen-Birkhoff [120, 121] theorem in TeVeS due to the presence of additional fields to the metric. Thus vacuum spherically symmetric solutions are not unique. For the above solution to describe a Black Hole, the candidate event horizon at $r = r_c$ must have bounded surface area (which implies $n \leq 2$) and not lead to an essential curvature singularity (which implies n = 2 or n > 4) [68]. These two conditions taken together imply that Black Holes in TeVeS require n = 2, in which case the metric above becomes exactly the Schwartzschild metric. This also allows us to determine M_s in terms of r_c , μ_0 and K from (47).

There is one caveat that results from this solution, that is for small enough r but still greater than r_c , the scalar field can acquire negative values, i.e. since $\frac{r-r_c}{r+r_c} < 1$ for all $r > r_c$, there is always some radius $r_1(\phi_c)$ such that $\phi < 0$ for $r < r_1$. As shown by Bekenstein [37] this is sufficient to lead to superluminal propagation of perturbations when viewed with the physical metric. Thus very close to the Black Hole it would be possible to create closed signal curves.

It would thus seem that Black Holes in TeVeS are unphysical. However, in the light of the absence of a Jebsen-Birkhoff theorem, there is also the possibility that there exists a different Black Hole solution which does not allow for the scalar field to become negative and is thus physical.

Indeed this other solution branch has been found by Sagi and Bekenstein as a byproduct while studying charged Black Hole solutions [68]. Sagi and Bekenstein made an educated guess that a charged Black Hole in TeVeS has the Reissner-Nordström physical metric,

$$ds^{2} = -\frac{(r^{2} - r_{h}^{2})^{2}}{(r^{2} + G_{N}Mr + r_{h}^{2})^{2}}dt^{2} + \frac{(r^{2} + G_{N}Mr + r_{h}^{2})^{2}}{r^{4}}(dr^{2} + r^{2}d\Omega^{2})$$
(50)

where M is the physical mass of the black hole and not the mass-parameter used in the original work [68], r_h is a scale given by $4r_h^2 = G_N(G_NM^2 - Q^2)$ and Q is the physical Black Hole charge *. Remember that the observed Newton's constant G_N

^{*} It might not be directly apparent that M and Q are the physical mass and charges of the Black hole. It is not hard, however, to show that they are as was done by Sagi and Bekenstein [68].

is related to G through (22). Note also that I'm using rescaled units from [68] such that no e^{ϕ_c} appear in the physical metric. This is done by the usual transformation $t \to e^{-\phi_c}t$ and $r \to e^{\phi_c}r$ (thus one has to also transform all relevant dimensionful variables e.g. $r_h \to e^{\phi_c}r_h$).

The above metric uniquely determines the solution for the electic field as usual. This enabled them to find solutions for the scalar field and for the vector field under the assumption that it points to the time direction. It turns out that the scalar field equation permits two branches of solutions. One of them, corresponds to the Giannios solution in the $Q \to 0$ limit which allows for superluminal propagation, a conclusion also unchanged in the charged case. The other branch, however, does not have the problem of superluminality. In that case the scalar field is given by

$$\phi = \phi_c + \delta_+ \ln(1 + \frac{r_h}{r}) + \delta_- \ln(1 - \frac{r_h}{r}) - \frac{1}{2}(\delta_+ + \delta_-) \ln(1 + \frac{G_N M}{r} + \frac{r_h^2}{r^2})$$
(51)

where the two constants δ_{+} and δ_{-} are given by

$$\delta_{\pm} = \frac{2 - K \pm \sqrt{2(2 - K) + \mu_0 K}}{2 - K + \mu_0} \tag{52}$$

Clearly $\delta_+ \geq 0$ and $\delta_- \leq 0$, therefore the term $\delta_- \ln(1 - \frac{r_h}{r})$ is always positive which means that for $\phi_c > 0$ (which would also make sense cosmologically) we have $\phi > 0$ and superluminal propagation is always avoided close to the black hole. Away from the black hole the situation can change. Expanding (51) to $O(\frac{1}{r})$ as $r \to \infty$ we get that ϕ approaches the asymptote ϕ_c from above iff

$$\frac{G_N M}{2r_h} \le \frac{\delta_+ - \delta_-}{\delta_+ + \delta_-} = \frac{2\sqrt{2(2 - K) + \mu_0 K}}{2 - K} \tag{53}$$

which is the condition found in [68] in a different manner. Thus if the condition above is satisfied then superluminal propagation is avoided everywhere for $\phi_c > 0$. On the other hand violation of the above inequality can lead to superluminal propagation unless a sufficiently large value for ϕ_c is used. Since we expect ϕ_c to be small, this limits the possibilities. In particular for extremal Black Holes, for which $r_h = 0$ we can never satisfy the above condition.

Taking the limit $Q \to 0$ (for which $G_N M = 2r_h = 2r_c$) we get the Giannios solution but with a negative M_s . If the above solution was the exterior solution to a star then it would have been discarded. Since however it is a black hole, M_s is no longer determined by (48) but only the ratio $\delta_- = \frac{2}{\mu_0} \frac{G}{G_N} \frac{M_s}{M} \le 0$ which is determined in terms of the TeVeS parameters is important. As a consequence, it is not clear how such a black hole (even when Q = 0) could arise from gravitational collapse, and this could be the subject of a future investigation.

Sagi and Bekenstein conclude their paper with the thermodynamics of these charged Black Holes and find that the usual Black Hole theormodynamics for the physical metric are recovered in TeVeS. It has been shown by Dubovsky and Sibiryakov [122] that theories with Lorentz symmetry breaking via a time-dependent scalar field, in which there is more than one maximal propagation speed, could lead to perpetual motion machines. It was also found by Eling et al [123] that classical violation of the 2nd law of thermodynamics is also possible in Einstein-Æther theory, also relying on two maximal propagation speeds. Given that TeVeS has both of these characteristics, i.e. local Lorentz symmetry violation and different maximal

propagation speeds for electromagnetic and gravitational waves, does it give rise to these problems? Sagi and Bekenstein answer this question to the negative.

5.1.2. Violation of the Jebsen-Birkhoff theorem in TeVeS. It is obvious from the different families of vacuum solutions found in TeVeS theory that the Jebsen-Birkhoff theorem is violated. In a way this was to be expected due to the presence of additional gravitational fields to the metric. As shown by Dai, Matsuo and Starkman [124], the consequences of the violation of the Jebsen-Birkhoff theorem can be dramatic and can complicate the observability of such theories. This means there are no truly isolated objects in relativistic MOND theories as even a small perturbation can lead to large violations of the theorem. Further studies should be conducted, however, to establish the extend of which this can influence observations in TeVeS theory and related variants.

5.1.3. Neutron stars Jin and Li considered the Tolman-Oppenheimer-Volkoff [125, 126] equation in the context of TeVeS under the assumption of strong-field limit $\mu = \mu_0$ and constant energy density inside the star [127] and solved the equations perturbatively in terms of the distance from the centre of the star.

Lasky, Sotani and Giannios made a much more thorough study, by considering neutron stars [128], again under the assumption $\mu = \mu_0$ (strong-field limit). They numerically solved the Tolman-Oppenheimer-Volkoff equations for hydrostatic equilibrium inside a spherically symmetric star with the realistic polytropic equation of state of Pandharipande [129] (equation of state "A" of Arnett and Bowers [130]). They further assumed that both the vector field and fluid velocity are aligned with the time direction and are thus proportional ($u_a = e^{\phi}A_a$) in which case the vector field equations are identically satisfied. Not surprisingly they find that sensible solutions are possible only if K < 2, which is indeed what is implied by positivity of energy and positivity of the effective Newton's constant (22).

The interior neutron star solution is matched to the exterior Schwartzschild-TeVeS solution found by Giannios [119] which is then used to define an Arnowitt-Deser-Misner (ADM) mass for the star. This last assumption, however, is a point that should warrant further investigation in the future. As discussed above, the Schwartzschild-TeVeS solution is valid only in the $\mu=\mu_0$ limit and thus it is correct only close to a black hole, or the surface of the star. Far away from the star, as the potential gradient drops, the solution would eventually move away from $\mu=\mu_0$ and, in a way that would depend on the TeVeS function $V(\mu)$, settle towards the MOND regime. The exception to this would be if the star is situated in an external field such as a galaxy, which itself is in the Newtonian regime. The MOND regime cannot lead to an asymptotically flat solution, at least not for the matter-frame metric, and thus the meaning of the ADM mass is questionable. Nevertheless, it is still a parameter that can be used to study these solutions.

By varying the two TeVeS parameters, K and μ_0 , they determined how the ADM mass parameter depends on the radius of the star. Both the ADM mass and the radius of the star can be significantly different from the ones in General Relativity. Taking an indicating upper bound for the masses of neutron stars as $M \sim 1.5 M_{\odot}$, they find that K is conservatively constrained to be at least less than unity, which is greater than typical values required by cosmology. They also consider the variation of the scalar field inside the star and find that depending on its initial cosmological

value ϕ_c , it can become negative in the interior. As found by Bekenstein [37] such a situation leads to superluminal propagation of perturbations which could be used to create closed signal curves [131]. This can be avoided provided $\phi_c > 10^{-3}$.

In the final section of the paper, the authors discuss possible observational signatures. They note that the redshift of atomic spectral lines emanating from the surface of the star depends on the TeVeS parameters. As discussed by DeDeo and Psaltis [132], measurements of these lines could in turn place constraints on gravitational theories (and therefore TeVeS theories and their variants) using X-ray observatories such as Chandra [133] and XMM-Newton [134]. A second test that Lasky, Sotani and Giannios propose is to probe TeVeS with gravitational wave astronomy, by determining the compactness of neutron stars through the use of w-mode oscillations [135], as these are almost independent of the equation of state of the neutron star [136].

As they also point out, studies of perturbations would also lead to predictions for quasi-periodic oscillations of neutron stars, which has also been proposed by DeDeo and Psaltis [137] as a possible test of gravitational theories. The quasi-periodic oscillations in the giant flares produced by magnetars (magnetized neutron stars) [138, 139, 140] seem to be in good aggreement with General Relativity [141], and would therefore provide strong tests for TeVeS and similar theories. Further studies of neutron stars could also involve adding a radial part to the vector field or choosing a different equation of state.

In a different study, Sotani [142] has analyzed the oscillation frequencies of neutron stars in TeVeS in the Cowling approximation. That is, he assumed that the fluctuations of the three gravitational fields \tilde{g}_{ab} , ϕ and A_a are frozen and considered only the fluctuations of the fluid. He then considered two different equations of state and determined the dependence of the frequency of the f-mode oscillations (fundamental modes) as a function of the stellar averaged density. It turns out that the result depends predominantly on the TeVeS constant K and that the dependence on the equation of state is weak. When K increases, the frequency of oscillation increases with larger average stellar masses. Observing such oscillations can then put constraints on the constant K.

5.1.4. Stability of spherically symmetric perturbations. Related to the spherically symmetric solutions discussed above is the issue of their stability to spherically symmetric perturbations. Seifert and Wald have formulated a variational method, based on earlier work of Wald [143], for determining the stability of spherically symmetric perturbations around a background spherically solution for any metric-based theory of gravity [144]. Seifert applied [145] this method to TeVeS, amongst other theories. He found that in the case of TeVeS the variational method is inapplicable because it does not lead to a well defined inner product on the space of perturbational fields. Using a WKB approximation, however, he was able to show that the branch of the Schwartzschild-TeVeS solution found by Giannios [119], where the vector field is aligned with the time direction, is unstable.

The question of stability of other solutions, such as the charged black hole and the 2nd branch of the Schwartzschild-TeVeS solution found by Sagi and Bekenstein [68], or the neutron star solutions found by Lasky, Sotani and Giannios [128] is still open. It is also an open question whether other TeVeS variants could provide for stable solutions. This is certainly a possibility, as Seifert has demonstrated that parameter spaces exist in general Einstein-Æther theories for which the spherically symmetric

solutions are stable. Moreover as shown by Zlosnik [146] vacuum solutions in a pure AQUAL theory are stable. These issues are currently under investigation [147].

Before concluding this section, a brief comment is in order. Whether some solutions are unstable does not necessarily mean that there are no stable solutions but rather that the unstable solution cannot be realized in nature. The issue of the existence of stable solutions is a different matter all together.

5.2. General stability and gravitational collapse

Contaldi, Wiseman and Withers [148] investigated the gravitational collapse of boson stars in TeVeS. They found that quite generically, the present TeVeS action leads to caustic formation and naked singularities before a Black Hole horizon can form. They trace the source of the caustic formation to the vector field and the particular form of its action. This is a serious theoretical obstacle to TeVeS. However, as they point out, slight modifications of the vector field action can introduce terms which prevent caustic formation in the absence of the scalar field.

A full investigation of caustic formation with general vector field action in TeVeS and with the full inclusion of the scalar field is still lacking. Their results show that it is an important issue to be addressed.

5.3. Parameterized Post-Newtonian parameters and tests of TeVeS in the solar system

The first calculation of the PPN parameter $\beta^{(PPN)}$ and $\gamma^{(PPN)}$ was done by Bekenstein [37], where he showed that $\gamma^{(PPN)} = 1$ and $\beta^{(PPN)} = 1 \sharp$. Thus TeVeS gives identical predictions to General Relativity for these two parameters.

The (almost) full set of PPN parameters were calculated by Tamaki [150] using Will's procedure [66, 151]. He verified Bekenstein's and Giannios's calculation of $\beta^{(PPN)}$ and $\gamma^{(PPN)}$, and further found that $\xi^{(PPN)} = \zeta_i^{(PPN)} = \alpha_3^{(PPN)} = 0$. His method, however, did not allow him to calculate the very important prefered frame parameters $\alpha_1^{(PPN)}$ and $\alpha_2^{(PPN)}$. The complete set of PPN parameters (including $\alpha_1^{(PPN)}$ and $\alpha_2^{(PPN)}$) were finally calculated by Sagi [152].

The PPN parameters determine corrections to Newtonian gravity towards the strong-field regime and cannot determine deviations towards the MOND regime. These deviations could be important and detectable in the solar system. For example, one place where such deviations could play a role is the Pioneer anomaly [71]. Such possibilites are discussed by Sanders [153]. Calculating MOND corrections in TeVeS theory (and indeed in any relativistic MOND theory) is still an open problem, although a first attempt was initiated by Bonvin et al. [154] for the case of the generalized Einstein-Æther theory. Further solar system tests of relativistic MOND theories could also be performed using different techniques [155].

5.4. Gravitational lensing

Gravitational lensing has become a crucial observational tool in modern Cosmology. It is often cited as giving the most compelling evidence for the existence of Dark Matter. Objects like the Bullet cluster rely on both strong and weak gravitational lensing to

The $\beta^{(PPN)}$ parameter in [37] is incorrect. The correct $\beta^{(PPN)}$ can by found in the erratum [149] with thanks to Giannios (see [119]).

map the gravitational potential wells to show that they are displaced from the majority of the visible baryon mass. It is thus of particular importance to determine the TeVeS predictions on gravitational lensing.

A first calculation of the deflection of light in TeVeS has been done by Bekenstein [37]. He showed that in the weak acceleration regime TeVeS provides the right deflection of light as if dark matter was present. The vector field plays a crucial role in the derivation through the disformal transformation. Indeed that was precisely the reason for introducing the vector field into the theory in the first place by Sanders [35]. Various authors have tested TeVeS with gravitational lenses and a short introduction of gravitational lensing in TeVeS can be found in [156].

Theoretical aspects of gravitational lensing in TeVeS have been discussed by Chiu, Ko and Tian [157] who looked at pointlike masses. They point out that the difference in amplifications for two images coming from a distant source is no longer unity as in GR, and can depend on the masses. Zhao et al study a sample of double-image lenses from the CASTLES catalog [158] by modelling the lense as a point mass and with the Hernquist profile. Chen and Zhao [159] and Chen [160] calculated the probability of two images occurring as a function of their separation.

Feix, Fedeli and Bartelmann analyzed the effects of asymmetric systems on gravitational lensing in the non-relativistic approximation of TeVeS [161]. They used a Laurant expansion of the free function and found a strong dependence of the lensing properties on the extend of the lense along the line-of-sight. They found that each of their simulated TeVeS convergence maps had a strong resemblance with the dominant baryonic component. As a consequence they showed (in accordance with [162]) that TeVeS cannot explain the weak lensing map of the "Bullet Cluster" [7] without an additional dark component.

Xu et al studied the effect on large filaments on gravitational lensing in MOND and in the non-relativistic approximation of TeVeS [163]. They found that in contrast to the case of General Relativity, even if the projected matter density is zero one still gets image distorsion and magnification effects. They conclude that since galaxies and galaxy clusters reside in such filaments or are projected on such structures, it complicates the interpretation of the weak lensing shear map in TeVeS. Thus, as they argue, filamentary structures might contribute in a significant and complex fashion in the context of TeVeS in cases such as the "Bullet Cluster" [7] and "Cosmic wreck train" Abel520 [164].

Shan, Feix, Famaey and Zhao [165] fitted fifteen double-image lenses from the CASTLES catalog using the quasistatic non-relativistic approximation of TeVeS and the same free function as in [166]. They find good fits for ten double-image lenses, however, the remaining five lenses do not provide a resonable fit. They note that all of those five lenses are residing in or close to groups or clusters of galaxies. Since lensing in TeVeS and more generally MOND is much more sensitive to the three-dimensional distribution of the lens and of the environment than in General Relativity [163], non-linear effects could be important.

Chiu, Tian and Ko [167] performed a study of ten double-image lenses from the CASTLES catalog and twenty two lenses from the SLACS catalog. They find good agreement of TeVeS with lensing.

Finally Mavromatos, Sakellariadou and Yusaf [168] performed the first lensing calculation in TeVeS which explicitly involves solving the full relativistic equations with the vector field. This was done under the assumption that the vector field is aligned with the time direction and has no spatial component. They used lenses from

the CASTLES catalog and the specific choice of free function of Angus, Famaey and Zhao [162] parameterized by a single parameter. They found that for this free function, the choice of the parameter which leads to acceptable lensing without Dark Matter leads to very bad fits for galactic rotation curves, while the choice which gives good galactic rotation curves cannot explain lensing without additional dark matter. It may be, however, that having a spatial component in the vector field, having a different vector field action, or using a different free function could provide for the right lensing without compromising the galactic rotation curves.

5.5. Superluminality, causality and gravi-Cherenkov radiation

As shown by Elliott, Moore and Stoica [169], the Einstein-Æther theory would lead to gravi-Cherenkov radiation, unless the speed of the spin-0 and spin-1 modes of the vector field is superluminal. If this was also true in TeVeS it would be potential trouble. Furthermore how do we reconcile this statement with the requirement of the absence of closed signal curves in the physical frame?

TeVeS could provide a solution, since the production of gravi-Cherenkov radiation should be evaluated in the diagonal frame where the three gravitational fields decouple. It is in that frame that the speeds of all non-tensor modes should be superluminal. However it might still be that the speeds in the physical frame are subluminal.

Bruneton [170] analyzed the issue of sub/superluminality in theories such as TeVeS by studying the initial value problem. He found that superluminality need not create problems for the initial value formulation. As he showed, however, in the MOND limit $\mu \to 0$ the scalar field becomes non-propagating. Thus if the MOND limit is exact, there can be a singular surface surrounding each galaxy on which the scalar field does not propagate.

5.6. Time travel of gravitational waves

Gravitational waves in TeVeS have a different light-cone than electromagnetic waves or other massless particles due to the disformal transformation. The speed of gravity is thus expected to be different than the speed of light by factors of $e^{-2\phi_c}$.

Kahya and Woodard [171] (see also [172]) have used this difference to propose a test for TeVeS and other theories with the same feature. They propose to look at the time of arrival of gravitational waves and of neutrinos from distant supernovae. This was further studied by Desai, Kahya and Woodard in [173].

5.7. Binary pulsars

An important test on TeVeS is the timing of binary pulsars. Binary pulsars have provided strong tests for General Relativity. Any other theory which aims to replace General Relativity should be able to obey the Binary Pulsar constraints. So far TeVeS has not been tested with Binary Pulsars which remain an open problem.

6. How TeVeS was constructed

Now that TeVeS theory has been described, we proceed to analyse its features.

6.1. Scalar field

The action for the scalar field traces its roots to the Aquadradic Lagrangian theory of Bekenstein and Milgrom [31], named AQUAL. AQUAL is a casting of MOND into a proper, consistent, non-relativistic gravitational theory which effectively gives back the MOND law. Let us describe the transition from MOND to AQUAL in such a way as to prepare the discussion for relativistic MOND.

6.1.1. AQUAL A generic non–relativistic gravitational theory can be built as follows. We define two potentials, namely a Poisson potential Φ_P such that it obeys the Poisson equation, and a Newtonian potential Φ_N which obeys Newton's 2nd law of motion. More explicitly we have

$$\nabla^2 \Phi_P = 4\pi G_N \rho, \tag{54}$$

where G_N is Newton's constant and ρ is the total matter density, and

$$\vec{a} = -\nabla \Phi_N \tag{55}$$

where \vec{a} is the acceleration of a test body due to the Newtonian potential. To complete the theory, a relation between Φ_N and Φ_P must be given. In the simplest case, Newtonian gravity, the two potentials are equal: $\Phi_N = \Phi_P$.

Milgrom breaks the equality of the two potentials, by modifying Newton's 2nd law of motion (55) into

$$\mu_m(|\vec{a}|/a_0)\vec{a} = -\nabla\Phi_P \tag{56}$$

where $\mu_m(x) \to 1$ as $x \gg 1$ where one recovers the Newtonian limit, and $\mu_m(x) \to x$ as $x \ll 1$ which is the ultra MOND limit. In the ultra MOND limit we then have $|\vec{a}|^2 = a_0 |\nabla \Phi_P|$, which in the spherically symmetric case (with $\Phi_P \propto 1/r$ and $v^2 \propto |\vec{a}|r$) gives constant velocity curves. The function μ_m is left free with only the two limits completely specified.

We now pass to AQUAL. We see that in the MOND case, the two potentials are implicitely related through

$$\mu_m(|\nabla \Phi_N|/a_0)\nabla \Phi_N = \nabla \Phi_P \tag{57}$$

Specification of AQUAL now becomes trivial. One completely eliminates any reference to the Poisson potential and writes the gravitational equations as

$$\vec{\nabla} \cdot \left[\mu_m(|\vec{\nabla}\Phi_N|/a_0) \vec{\nabla}\Phi_N \right] = 4\pi G_N \rho \tag{58}$$

which together with (55) fully specify the theory. This equation is derivable from a Lagrangian with aquadratic kinetic term. The theory considered this way satisfies all conservation laws like conservation of energy, momentum and angular momentum.

6.1.2. Relativistic AQUAL In a relativistic theory, the gravitational potential is effectively defined as a small fluctuation of the gravitational metric, through the weak field limit. This suggests that the relativistic analog of AQUAL should be a bimetric theory. In other words, the relativistic analog of the Poisson potential Φ_P should be a metric \tilde{g}_{ab} and the analog of the Newtonian potential should be a metric g_{ab} . Furthermore, Newton's 2nd law is contained in the weak-field limit of the geodesic equation for g_{ab} . It is therefore clear that g_{ab} should be the universally coupled metric. On the other hand, the Poisson equation should be the weak-field limit of the Einstein equations for \tilde{g}_{ab} . We thus start from the prototype action $S = S_{\tilde{q}} + S_m$,

where $S_{\tilde{g}}$ is the Einstein-Hilbert action (4) for \tilde{g}_{ab} and S_m is the matter action which only contains g_{ab} . To complete the theory we must specify a relation between \tilde{g}_{ab} and g_{ab} .

Bekenstein and Milgrom [31] introduce a scalar field ϕ . The action for \tilde{g}_{ab} and ϕ is given as

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} \tilde{R} - \frac{a_0^2 (1+\beta_0)^2 \beta_0}{8\pi G} \int d^4x \sqrt{-\tilde{g}} f(X)$$
 (59)

where a_0 is Milgrom's constant, β_0 is another constant and f(X) is a free function. The variable X is given by $X = \frac{\tilde{g}^{ab}\nabla_a\phi\nabla_b\phi}{a_0^2(1+\beta_0)^2}$ They complete the theory by imposing a conformal relation between \tilde{g}_{ab} and g_{ab} as $g_{ab} = e^{2\phi}\tilde{g}_{ab}$. Notice that this theory has virtually all of TeVeS features except the inclusion of the vector field. Its quasistatic limit is effectively the same as for TeVeS and reproduces the MOND phenomenology. Bekenstein and Milgrom showed, however, that the fluctuations of ϕ propagate acausally when viewed with the universally coupled metric g_{ab} . This is a general consequence of f(X) having a MOND limit.

6.1.3. Phase-coupling gravitation In an attempt to remove the acausal propagation in the relativistic AQUAL theory, Bekenstein invented the Phase-coupling gravitational theory [33]. In this theory one replaces the real scalar field above by a complex scalar field χ which has a conventional action with respect to the metric \tilde{g}_{ab} , with a canonical kinetic term and a potential. The real scalar field ϕ is identified with the phase of χ as $\chi = |\chi|e^{i\phi}$ and is still assumed to relate the two metrics by a conformal transformation $g_{ab} = e^{2\eta_p\phi}\tilde{g}_{ab}$ where η_p is a parameter. In the non-relativistic limit one finds an effective AQUAL equation for ϕ due to the dependence of the amplitude $|\chi|$ on $\nabla \phi$ through the amplitude's field equation. All fluctuations are causally propagated in this theory, however, for the parameter values required by MOND phenomenology, this theory violates solar system constraints.

6.1.4. Disformal relativistic AQUAL There is one other problem associated with all of the above theories, that is the bending of light. It is straightforward to show that the bending of light associated with those theories will not be enough to account for observations without dark matter. The problem lies in the fact that the conformal relation does not change the lightcone structure.

Based on Bekenstein's investigation of the relation between physical and gravitational geometry [60], Bekenstein and Sanders considered a disformal transformation of the form $g_{ab} = e^{2\phi}[A(I)\tilde{g}_{ab} + \ell_B^2B(I)\nabla_a\phi\nabla_b\phi]$ where A and B are two functions of the invariant $I = g^{ab}\nabla_a\phi\nabla_b\phi$. The presence of the additive tensor $\nabla_a\phi\nabla_b\phi$ in the transformation above implies that the lightcone structure is broken, i.e. the two lightcones of g_{ab} and \tilde{g}_{ab} do not coincide. However, even with this generalization, the bending of light turns out to be smaller than GR based on the visible mass alone, if the theory is to be causal. The main obstacle is the fact that in the quasistatic limit $\nabla_\mu\phi=(0,\vec\nabla\phi)$, i.e. it is purely spatial, which means that the relation between g_{00} and \tilde{g}_{00} is the same as with a conformal transformation.

6.2. The vector field

Sanders [35] solved the lensing problem by introducing a unit-timelike vector field A_a . The idea is to change the relation between g_{00} and \tilde{g}_{00} from the one based on

a conformal transformation. This would require an additive piece like the disformal transformation above, but which would remain intact in the quasistatic limit. It is clear that a transformation of the form $g_{ab} = A(\phi)\tilde{g}_{ab} + B(\phi)A_aA_b$ does the trick. Sanders fixed the form of A and B based on the requirement of invariance under global redefinition of units and the requirement that the fine-structure constant is independent of the frame of formulation of the theory (with g_{ab} or with \tilde{g}_{ab}). This leads uniquely to the disformal transformation in TeVeS (1). He further showed that the theory reproduces the right bending of light in a quasistatic situation.

Bekenstein [37] turned the Sanders's proposal into a fully relativistic theory by providing an action for A_a . He further modified the scalar field action such that the kinetic term is $\mu(\tilde{g}^{ab} - A^a A^b) \nabla_a \phi \nabla_b \phi$ rather than $\mu \tilde{g}^{ab} \nabla_a \phi \nabla_b \phi$ as in the relativistic AQUAL theory. This final ingredient was needed to ensure that the theory is is causal in all situations for which $\phi > 0$. Thus TeVeS was born.

7. Other TeVeS variants and spin-offs

7.1. Sanders Biscalar-Tensor-Vector theory

Sanders [174] considered a TeVeS variant which resembles the Phase-coupling gravitational theory. In this case the field μ is considered dynamical and has a kinetic term. Like Phase-coupling gravitation, the pair (μ, ϕ) can be combined into a complex scalar field $\mu e^{i\phi}$ which has a conventional action. The difference from Phase-coupling gravitation lies in the use of the vector field in the disformal transformation. MOND phenomenology in quasistatic situations follows but unlike Phase-coupling gravitation it does not require parameters which are in conflict with solar system tests. The theory also gives the right bending of light. Sanders also considers the possibility that the Milgrom's constant a_0 is given by the cosmological value of $\dot{\phi}$ by building the action out of the invariants $\tilde{g}^{ab}\nabla_a\phi\nabla_b\phi$ and $A^a\nabla_a\phi$.

7.2. The generalized Einstein-Aether theory

Zlosnik, Ferreira and Starkman [175] showed that one can eliminate the Einstein-frame metric altogether and write TeVeS theory using the physical metric alone. In the process, the scalar field is combined with the vector field, to define a new vector field which is dynamically timelike but not unit. Thus TeVeS is equivalent to a vector-tensor theory (see appendix Appendix B.2).

Building upon this observation, Zlosnik, Ferreira and Starkman consider the following simplification. They re-insert the unit-timelike constraint of the vector field into the physical frame, thus removing one degree of freedom. They further simplify the functions multiplying the vector field kinetic term into constants, and the resulting action becomes the Einstein-Æther theory. To recover MOND phenomenology, they depart from the original Einstein-Æther theory and use a function $\mathcal F$ to write the action as

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[R - M_{\mathcal{E}}^2 \mathcal{F} \left(\mathcal{K} \right) + \lambda (A^a A_a + 1) \right] + S_m[g] \quad (60)$$

where M_{E} is a mass scale and

$$\mathcal{K} = \left[c_1 g^{ac} g^{bd} + c_2 g^{ab} g^{cd} + c_3 g^{ad} g^{bc} \right] \nabla_a A_b \nabla_c A_d \tag{61}$$

thus this theory is called the generalized Einstein-Æther theory. In the weak field quasistatic limit one recovers MOND phenomenology, like TeVeS.

It has been shown that the theory can lead to structure formation in a similar manner to TeVeS [176]: the vector field plays a key role to sourcing potential wells. Cosmological studies of such theories have been conducted in the framework of the original Einstein-Æther theory [177, 178] while the confrontation of the generalized Einstein-Æther theory with cosmological observations is under way [179].

The theory has also been confronted with solar system tests [154] while gravitational lensing, in particular concerning the Bullet Cluster [7], has been studied by Dai, Matsuo and Starkman [180].

7.3. The generalized TeVeS theory

A simple generalization of TeVeS theory has been considered by Skordis [78]. In this case one generalizes the vector field action in to the same form as in the Einstein-Æther theory. In particular, we have

$$S_A = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[K^{abcd} \tilde{\nabla}_a A_b \tilde{\nabla}_c A_d + \lambda (A^a A_a + 1) \right]$$
 (62)

where the tensor K^{abcd} is given by

$$K^{abcd} = c_1 \tilde{g}^{ac} \tilde{g}^{bd} + c_2 \tilde{g}^{ab} \tilde{g}^{cd} + c_3 \tilde{g}^{ad} \tilde{g}^{bc} + c_4 A^a A^c \tilde{g}^{bd}$$
(63)

The coefficients c_i are related to the ones in [78] as $c_1 = K_B + K_+$, $c_2 = K_0$, $c_3 = -K_B + K_+$ and $c_4 = K_A$. This generalization preserves the MOND limit as was explicitly calculated by Contaldi, Wiseman and Withers [148].

Cosmology in this theory has similar features as in TeVeS [78]. It is easier to define new coefficients $K_t = K_B + K_+ - K_A$, $\kappa_d = K_+ + \frac{1}{2}K_0$, $K_F = 1 + K_0 + \kappa_d$ and $R_K = 1 - \frac{3\kappa_d}{K_F}$. At the background FLRW level, the vector field simply rescales the gravitational constant. In particular the Friedmann equation in the Bekenstein frame becomes

$$3K_F \tilde{H}^2 = 8\pi G(\rho_\phi + \rho) \tag{64}$$

and so one must have $K_F > 0$. Thus as for the background dynamics, only the constant K_F plays a role by simply rescaling G. Therefore all FLRW solutions to TeVeS theory would also be solutions in this general version [78].

At the linearized level, only three constants play a role for scalar perturbations, namely K_t , K_F and R_K . For vector perturbations all four constants are needed and for tensor perturbations only K_F and R_K are important. The primordial adiabatic mode for scalar perturbations has been constructed in [78] for this generalized case. The constant K_t plays the same role as K_t in TeVeS by allowing for a growing mode in the vector field in order to source structure formation. The constants K_F and R_K can also play a role and can introduce interesting features in the power spectra [181]. In particular $R_K \neq 1$ can introduce a damping term into the scalar mode α of the vector field equation α . If $R_K < 1$ then $\kappa_d > 0$ which damps the vector field perturbations and can stop their growth. If $R_K > 1$ then $\kappa_d < 0$ which introduces an instability and the vector field grows without bound.

7.4. The Halle-Zhao construction

Halle and Zhao [182] considered a generalization of TeVeS, the generalized Einstein-Æther theory and Zhao's $\nu\Lambda$ theory [183]. They propose vector-tensor theory in a single frame where the vector field action is a generalized function of both the vector

field kinetic terms $\nabla_a A_b$ and the scalar $A_a A^a$ representing the magnitude of the vector field. The vector field can be dynamically timelike or by a constraint. The different theories above come out as special cases.

8. Outlook

I have reviewed the Tensor-Vector-Scalar theory which has been proposed as a relativistic realization of Modified Newtonian Dynamics. The theory was a product of past antecedent theories, namely the Aquadratic Lagrangian Gravity (AQUAL) and its relativistic version [31], the phase-coupling gravitation [33] the disformal relativistic scalar field theory [61] and the Sanders's stratified vector field theory [35]. Since its inception [37] it has generated a generous amount of research on various aspects, starting with cosmology [37, 72, 73, 74, 75, 76, 67, 77, 78, 79], spherically symmetric solutions [37, 119, 127, 68, 128], gravitational collapse and stability [145, 148], solar system tests [37, 119, 155, 150, 152], gravitational lensing [37, 157, 159, 166, 156, 160, 161, 163, 165, 167, 168] issues on superluminality [170] and the travel time of gravitational waves [171, 172, 173].

There are many open questions and problems still remaining to be settled in TeVeS theory. Perhaps the most looming problem awaiting solution is to reconcile TeVeS with observations of the CMB radiation. This may require inclusion of isocurvature modes, a different free function, or a different TeVeS action.

Another important problem is the issue of stability, in particular of spherically symmetric perturbations, and the avoidance of caustic singularities. This may also require changing the vector field action.

It is also of particular importance to resolve many issues surrounding gravitational lensing. These are issues of the environmental MOND effects on the lensing predictions, such as from filaments, which can obscure tests of TeVeS with gravitational lensing.

Finally, it would be important to resolve problems with clusters of galaxies and galaxy groups as well as the bullet-clusters. It particular it is vital to elucidate the role of the vector field on those scales. Since it has been shown that the vector field plays a fundamental role in driving large scale structure formation, it should be deemed important on cluster scales and may be the key to solving the problems of those systems with MOND.

Further open questions include binary pulsar tests, isocurvature modes, cosmological evolution for other free functions, cosmic acceleration without an effective cosmological constant, solar system tests, and de Sitter black holes.

As the influx of cosmological data continues in the next few years, theories such as TeVeS should be considered further as an explanation of the missing mass and the missing energy problems. They provide interesting and promising research directions.

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9. Bibliography

- [1] J. Oort. The force exerted by the stellar system in the direction perpendicular to the galactic plane and some related problems. *Bull. Astron. Inst. Neth.*, 6:249, 1932.
- [2] F. Zwicky. Spectral displacement of extra galactic nebulae. Helv. Phys. Acta, 6:110-127, 1933.

- [3] Sinclair Smith. The Mass of the Virgo Cluster. Astrophys. J., 83:23, 1936.
- [4] Vera C. Rubin and Jr. Ford, W. Kent. Rotation of the Andromeda Nebula from a Spectroscopic Survey of Emission Regions. Astrophys. J., 159:379–403, 1970.
- [5] V. C. Rubin, N. Thonnard, and Jr. Ford, W. K. Rotational properties of 21 SC galaxies with a large range of luminosities and radii, from NGC 4605 (R = 4kpc) to UGC 2885 (R = 122kpc). Astrophys. J., 238:471, 1980.
- [6] Yoshiaki Sofue and Vera Rubin. Rotation Curves of Spiral Galaxies. Ann. Rev. Astron. Astrophys., 39:137–174, 2001. astro-ph/0010594.
- [7] Douglas Clowe et al. A direct empirical proof of the existence of dark matter. Astrophys. J., 648:L109–L113, 2006. astro-ph/0608407.
- [8] M. R. Nolta et al. Five-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Angular Power Spectra. Astrophys. J. Supp., 180:296–305, 2009. 0803.0593.
- [9] Brian P. Schmidt et al. The High-Z Supernova Search: Measuring Cosmic Deceleration and Global Cur vature of the Universe Using Type Ia Supernovae. Astrophys. J., 507:46-63, 1998. astro-ph/9805200.
- [10] S. Perlmutter et al. Measurements of Omega and Lambda from 42 High-Redshift Supernovae. Astrophys. J., 517:565–586, 1999. astro-ph/9812133.
- [11] C. Alcock et al. EROS and MACHO Combined Limits on Planetary-Mass Dark Matter in the Galactic Halo. Astrophys. J. Lett., 499:L9-L12, 1998.
- [12] C. Alcock et al. The MACHO project: Microlensing results from 5.7 years of LMC observations. Astrophys. J., 542:281–307, 2000. astro-ph/0001272.
- [13] C. Afonso et al. Limits on Galactic Dark Matter with 5 Years of EROS SMC Data. Astron. Astrophys., 400:951–956, 2003. astro-ph/0212176.
- [14] Jaiyul Yoo, Julio Chaname, and Andrew Gould. The End of the MACHO Era: Limits on Halo Dark Matter from Stellar Halo Wide Binaries. Astrophys. J., 601:311–318, 2004. astroph/0307437.
- [15] S. Eidelman et al. Review of particle physics. Phys. Lett., B592:1, 2004.
- [16] Gary Steigman. Neutrinos and big bang nucleosynthesis. Phys. Scripta, T121:142–146, 2005. hep-ph/0501100.
- [17] J. R. Bond and A. S. Szalay. The Collisionless Damping of Density Fluctuations in an Expanding Universe. Astrophys. J., 274:443–468, 1983.
- [18] S. S. Gershtein and Ya. B. Zeldovich. Rest mass of muonic neutrino and cosmology. JETP Lett., 4:120–122, 1966.
- [19] G. Marx and A. S. Szalay. in Neutrino 72', 1:191, 1972. Budapest: Technoinform.
- [20] A. S. Szalay and G. Marx. Neutrino rest mass from cosmology. Astron. Astrophys., 49:437–441, 1976.
- [21] R. Cowsik and J. McClelland. Gravity of neutrinos of nonzero mass in astrophysics. Astrophys. J., 180:7–10, 1973.
- [22] S. Tremaine and J. E. Gunn. Dynamical role of light neutral leptons in cosmology. Phys. Rev. Lett., 42:407–410, 1979.
- [23] V. M. Lobashev et al. Direct search for mass of neutrino and anomaly in the tritium betaspectrum. Phys. Lett., B460:227–235, 1999.
- [24] C. Weinheimer et al. High precision measurement of the tritium beta spectrum near its endpoint and upper limit on the neutrino mass. *Phys. Lett.*, B460:219–226, 1999.
- [25] C. Weinheimer et al. Erratum to: High precision measurement of the tritium beta spectrum near its endpoint and upper limit on the neutrino mass. Phys. Lett., B464:352, 1999.
- [26] K. Griest and M. Kamionkowski. Supersymmetric dark matter. Phys. Rept., 333:167–182, 2000.
- [27] Gianfranco Bertone, Dan Hooper, and Joseph Silk. Particle Dark Matter: Evidence, Candidates and Constraints. Phys. Rept., 405:279–390, 2005. hep-ph/0404175.
- [28] Edmund J. Copeland, M. Sami, and Shinji Tsujikawa. Dynamics of dark energy. Int. J. Mod. Phys., D15:1753–1936, 2006. hep-th/0603057.
- [29] Andreas Johann Albrecht et al. Report of the Dark Energy Task Force. astro-ph/0609591.
- [30] A. Finzi. On the validity of Newton's law at a long distance. Mon. Not. R. Astron. Soc., 127:21, 1963.
- [31] J. Bekenstein and Mordehai Milgrom. Does the missing mass problem signal the breakdown of Newtonian gravity? Astrophys. J., 286:7–14, 1984.
- [32] J. R. Kuhn and L. Kruglyak. Non-Newtonian forces and the invisible mass problem. Astrophys. J., 313:1–12, 1987.
- [33] J. D. Bekenstein. Phase coupling gravitation: symmetries and gauge fields. Phys. Lett., B202:497–500, 1988.

- [34] Philip D. Mannheim and Demosthenes Kazanas. Exact vacuum solution to conformal Weyl gravity and galactic rotation curves. Astrophys. J., 342:635–638, 1989.
- [35] R. H. Sanders. A stratified framework for scalar-tensor theories of Modified Dynamics. Astrophys. J., 480:492–502, 1997. astro-ph/9612099.
- [36] I. T. Drummond. Bimetric gravity and [dark matter]. Phys. Rev., D63:043503, 2001. astro-ph/0008234.
- [37] Jacob D. Bekenstein. Relativistic gravitation theory for the MOND paradigm. Phys. Rev., D70:083509, 2004. astro-ph/0403694.
- [38] J. W. Moffat. Scalar-tensor-vector gravity theory. JCAP, 0603:004, 2006. gr-qc/0506021.
- [39] Kirill Krasnov and Yuri Shtanov. Non-Metric Gravity II: Spherically Symmetric Solution, Missing Mass and Redshifts of Quasars. Class. Quant. Grav., 25:025002, 2008. 0705.2047.
- [40] Kirill Krasnov and Yuri Shtanov. Halos of Modified Gravity. Int. J. Mod. Phys., D17:2555–2562, 2009. 0805.2668.
- [41] Máximo Bañados. A Born-Infeld action for dark energy and dark matter. Phys. Rev., D77:123534, 2008. 0801.4103.
- [42] Maximo Banados, Andres Gomberoff, Davi C. Rodrigues, and Constantinos Skordis. A note on bigravity and dark matter. 0811.1270.
- [43] M. Milgrom. A Modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis. Astrophys. J., 270:365–370, 1983.
- [44] M. Milgrom. A Modification of the Newtonian dynamics: Implications for galaxies. Astrophys. J., 270:371–383, 1983.
- [45] M. Milgrom. A modification of the Newtonian dynamics: implications for galaxy systems. Astrophys. J., 270:384–389, 1983.
- [46] Mordehai Milgrom and Erez Braun. The rotation curve of DDO 154: A Particularly acute test of the modified dynamics. Astrophys. J., 334:130–134, 1988.
- [47] K. G. Begeman, A. H. Broeils, and R. H. Sanders. Extended rotation curves of spiral galaxies: Dark haloes and modified dynamics. Mon. Not. Roy. Astron. Soc., 249:523, 1991.
- [48] R. H. Sanders. The published extended rotation curves of spiral galaxies: Confrontation with Modified Dynamics. astro-ph/9606089.
- [49] W. J. G. de Blok and S. S. McGaugh. Testing Modified Newtonian Dynamics with Low Surface Brightness Galaxies –Rotation curve fits. astro-ph/9805120.
- [50] Stacy S. McGaugh and W. J. G. de Blok. Testing the Hypothesis of Modified Dynamics with Low Surface Brightness Galaxies and Other Evidence. Astrophys. J., 499:66–81, 1998. astro-ph/9801102.
- [51] Mordehai Milgrom and Robert H. Sanders. MOND predictions of halo phenomenology in disc galaxies. Mon. Not. Roy. Astron. Soc., 357:45–48, 2005. astro-ph/0406487.
- [52] Mordehai Milgrom and Robert H. Sanders. MOND rotation curves of very low mass spiral galaxies. astro-ph/0611494.
- [53] Stacy S. McGaugh, Jim M. Schombert, Greg D. Bothun, and W. J. G. de Blok. The Baryonic Tully-Fisher Relation. Astrophys. J., 533:L99–L102, 2000. astro-ph/0003001.
- [54] Stacy S. McGaugh. The Baryonic Tully-Fisher Relation of Galaxies with Extended Rotation Curves and the Stellar Mass of Rotating Galaxies. Astrophys. J., 632:859–871, 2005. astro-ph/0506750.
- [55] R. H. Sanders. Mass discrepancies in galaxies: dark matter and alternatives. Astron. Astrophys. Rev., 2:1–28, 1990.
- [56] Mordehai Milgrom. The modified dynamics: A status review. astro-ph/9810302.
- [57] Robert H. Sanders and Stacy S. McGaugh. Modified Newtonian Dynamics as an Alternative to Dark Matter. Ann. Rev. Astron. Astrophys., 40:263–317, 2002. astro-ph/0204521.
- [58] Mordehai Milgrom. The MOND paradigm. 0801.3133.
- [59] Jean-Philippe Bruneton and Gilles Esposito-Farese. Field-theoretical formulations of MOND-like gravity. Phys. Rev., D76:124012, 2007. 0705.4043.
- [60] Jacob D. Bekenstein. The Relation between physical and gravitational geometry. Phys. Rev., D48:3641–3647, 1993. gr-qc/9211017.
- [61] Jacob D. Bekenstein and Robert H. Sanders. Gravitational lenses and unconventional gravity theories. Astrophys. J., 429:480, 1994. astro-ph/9311062.
- [62] Jacob D. Bekenstein and Robert H. Sanders. A Primer to Relativistic MOND Theory. astro-ph/0509519.
- [63] Jacob D. Bekenstein. Modified gravity vs dark matter: Relativistic theory for MOND. PoS, JHW2004:012, 2005. astro-ph/0412652.
- [64] Jacob D. Bekenstein. Relativistic MOND as an alternative to the dark matter paradigm. 0901.1524.

- [65] Jacob D. Bekenstein. Modified Gravity as an Alternative to Dark Matter. Chapter 6 in Particle Dark Matter, ed. G. Bertone, Cambridge University Press, Cambridge (2009).
- [66] C. Will. Theory and experiment in gravitational physics. Cambridge University Press, Cambridge (1981).
- [67] F. Bourliot, P. G. Ferreira, D. F. Mota, and C. Skordis. The cosmological behavior of Bekenstein's modified theory of gravity. *Phys. Rev.*, D75:063508, 2007. astro-ph/0611255.
- [68] Eva Sagi and Jacob D. Bekenstein. Black holes in the TeVeS theory of gravity and their thermodynamics. Phys. Rev., D77:024010, 2008. 0708.2639.
- [69] Jacob D. Bekenstein and Eva Sagi. Do Newton's G and Milgrom's a₀ vary with cosmological epoch? Phys. Rev., D77:103512, 2008. 0802.1526.
- [70] Christopher Limbach, Dimitrios Psaltis, and Feryal Ozel. The Redshift Evolution of the Tully-Fisher Relation as a Test of Modified Gravity. 0809.2790.
- [71] Slava G. Turyshev, Michael Martin Nieto, and John D. Anderson. The Pioneer anomaly and its implications. EAS Publ. Ser., 20:243–250, 2006. gr-qc/0510081.
- [72] J. G. Hao and R. Akhoury. Can relativistic MOND theory resolve both the dark matter and dark energy paradigms? astro-ph/0504130.
- [73] Constantinos Skordis, D. F. Mota, P. G. Ferreira, and C. Boehm. Large scale structure in Bekenstein's theory of relativistic MOND. *Phys. Rev. Lett.*, 96:011301, 2006. astro-ph/0505519.
- [74] Luz Maria Diaz-Rivera, Lado Samushia, and Bharat Ratra. Inflation and accelerated expansion TeVeS cosmological solutions. Phys. Rev., D73:083503, 2006. astro-ph/0601153.
- [75] Scott Dodelson and Michele Liguori. Can Cosmic Structure form without Dark Matter? Phys. Rev. Lett., 97:231301, 2006. astro-ph/0608602.
- [76] Constantinos Skordis. TeVeS cosmology: Covariant formalism for the background evolution and linear perturbation theory. Phys. Rev., D74:103513, 2006. astro-ph/0511591.
- [77] HongSheng Zhao. Constraining TeVeS gravity as effective dark matter and dark energy. Int. J. Mod. Phys., D16:2055–2063, 2008. astro-ph/0610056.
- [78] Constantinos Skordis. Generalizing TeVeS Cosmology. Phys. Rev., D77:123502, 2008. 0801.1985.
- [79] P. G. Ferreira, C. Skordis, and C. Zunckel. Dark Matter, Modified Gravity and the Mass of the Neutrino. Phys. Rev., D78:044043, 2008. 0806.0116.
- [80] C. Armendariz-Picon, T. Damour, and Viatcheslav F. Mukhanov. k-Inflation. Phys. Lett., B458:209-218, 1999. hep-th/9904075.
- [81] C. Armendariz-Picon, Viatcheslav F. Mukhanov, and Paul J. Steinhardt. A dynamical solution to the problem of a small cosmological constant and late-time cosmic acceleration. *Phys. Rev. Lett.*, 85:4438–4441, 2000. astro-ph/0004134.
- [82] Christof Wetterich. The Cosmon model for an asymptotically vanishing time dependent cosmological 'constant'. Astron. Astrophys., 301:321–328, 1995. hep-th/9408025.
- [83] Pedro G. Ferreira and Michael Joyce. Structure formation with a self-tuning scalar field. Phys. Rev. Lett., 79:4740–4743, 1997. astro-ph/9707286.
- [84] Pedro G. Ferreira and Michael Joyce. Cosmology with a Primordial Scaling Field. Phys. Rev., D58:023503, 1998. astro-ph/9711102.
- [85] Andreas Albrecht and Constantinos Skordis. Phenomenology of a realistic accelerating universe using only Planck-scale physics. Phys. Rev. Lett., 84:2076–2079, 2000. astro-ph/9908085.
- [86] T. Barreiro, Edmund J. Copeland, and N. J. Nunes. Quintessence arising from exponential potentials. Phys. Rev., D61:127301, 2000. astro-ph/9910214.
- [87] Luca Amendola. Coupled quintessence. Phys. Rev., D62:043511, 2000. astro-ph/9908023.
- [88] Constantinos Skordis and Andreas Albrecht. Planck-scale quintessence and the physics of structure formation. Phys. Rev., D66:043523, 2002. astro-ph/0012195.
- [89] Domenico Tocchini-Valentini and Luca Amendola. Stationary dark energy with a baryon dominated era: Solving the coincidence problem with a linear coupling. *Phys. Rev.*, D65:063508, 2002. astro-ph/0108143.
- [90] Tomi S. Koivisto. Disformal quintessence. 0811.1957.
- [91] Martin Bucher, Kavilan Moodley, and Neil Turok. The general primordial cosmic perturbation. Phys. Rev., D62:083508, 2000. astro-ph/9904231.
- [92] M. Bucher, Joanna Dunkley, P. G. Ferreira, K. Moodley, and C. Skordis. The initial conditions of the universe: how much isocurvature is allowed? *Phys. Rev. Lett.*, 93:081301, 2004. astro-ph/0401417.
- [93] K. Moodley, M. Bucher, Joanna Dunkley, P. G. Ferreira, and C. Skordis. Constraints on isocurvature models from the WMAP first- year data. *Phys. Rev.*, D70:103520, 2004. astroph/0407304.

- [94] Joanna Dunkley, M. Bucher, P. G. Ferreira, K. Moodley, and C. Skordis. Measuring the geometry of the Universe in the presence of isocurvature modes. *Phys. Rev. Lett.*, 95:261303, 2005. astro-ph/0507473.
- [95] Patrick Crotty, Juan Garcia-Bellido, Julien Lesgourgues, and Alain Riazuelo. Bounds on isocurvature perturbations from CMB and LSS data. Phys. Rev. Lett., 91:171301, 2003. astro-ph/0306286.
- [96] Hannu Kurki-Suonio, Vesa Muhonen, and Jussi Valiviita. Correlated Primordial Perturbations in Light of CMB and LSS Data. Phys. Rev., D71:063005, 2005. astro-ph/0412439.
- [97] Maria Beltran, Juan Garcia-Bellido, Julien Lesgourgues, and Matteo Viel. Squeezing the window on isocurvature modes with the Lyman- alpha forest. *Phys. Rev.*, D72:103515, 2005. astro-ph/0509209.
- [98] Rachel Bean, Joanna Dunkley, and Elena Pierpaoli. Constraining Isocurvature Initial Conditions with WMAP 3- year data. Phys. Rev., D74:063503, 2006. astro-ph/0606685.
- [99] Roberto Trotta. The isocurvature fraction after WMAP 3-year data. Mon. Not. Roy. Astron. Soc. Lett., 375:L26–L30, 2007. astro-ph/0608116.
- [100] Masahiro Kawasaki and Toyokazu Sekiguchi. Cosmological Constraints on Isocurvature and Tensor Perturbations. Prog. Theor. Phys., 120:995–1016, 2008. 0705.2853.
- [101] D. F. Mota, P. G. Ferreira, and C. Skordis. Cosmic Microwave Background in TeVeS theory. Work presented at the Alternative gravities workshop in Edinburgh, 2006.
- [102] C. Skordis. In preparation.
- [103] Anze Slosar, Alessandro Melchiorri, and Joseph Silk. Did Boomerang hit MOND? Phys. Rev., D72:101301, 2005. astro-ph/0508048.
- [104] D. N. Spergel et al. Wilkinson Microwave Anisotropy Probe (WMAP) three year results: Implications for cosmology. Astrophys. J. Suppl., 170:377, 2007. astro-ph/0603449.
- [105] Maximo Banados, Pedro G. Ferreira, and Constantinos Skordis. Eddington-Born-Infeld gravity and the large scale structure of the Universe. 0811.1272.
- [106] Garry W. Angus. Are sterile neutrinos consistent with clusters, the CMB and MOND? 0805.4014.
- [107] Arthur Lue, Roman Scoccimarro, and Glenn D. Starkman. Probing Newton's constant on vast scales: DGP gravity, cosmic acceleration and large scale structure. *Phys. Rev.*, D69:124015, 2004. astro-ph/0401515.
- [108] Edmund Bertschinger. On the Growth of Perturbations as a Test of Dark Energy. Astrophys. J., 648:797–806, 2006. astro-ph/0604485.
- [109] Robert Caldwell, Asantha Cooray, and Alessandro Melchiorri. Constraints on a New Post-General Relativity Cosmological Parameter. Phys. Rev., D76:023507, 2007. astro-ph/0703375.
- [110] O. Dore et al. Testing Gravity with the CFHTLS-Wide Cosmic Shear Survey and SDSS LRGs. 0712.1599.
- [111] Bhuvnesh Jain and Pengjie Zhang. Observational Tests of Modified Gravity. 0709.2375.
- [112] Fabian Schmidt, Michele Liguori, and Scott Dodelson. Galaxy-CMB Cross-Correlation as a Probe of Alternative Models of Gravity. Phys. Rev., D76:083518, 2007. 0706.1775.
- [113] Pengjie Zhang, Michele Liguori, Rachel Bean, and Scott Dodelson. Probing Gravity at Cosmological Scales by Measurements which Test the Relationship between Gravitational Lensing and Matter Overdensity. Phys. Rev. Lett., 99:141302, 2007. 0704.1932.
- [114] Scott F. Daniel, Robert R. Caldwell, Asantha Cooray, and Alessandro Melchiorri. Large Scale Structure as a Probe of Gravitational Slip. Phys. Rev., D77:103513, 2008. 0802.1068.
- [115] Wayne Hu and Ignacy Sawicki. A Parameterized Post-Friedmann Framework for Modified Gravity. Phys. Rev., D76:104043, 2007. 0708.1190.
- [116] Wayne Hu. Parametrized Post-Friedmann Signatures of Acceleration in the CMB. Phys. Rev., D77:103524, 2008. 0801.2433.
- [117] Constantinos Skordis. Consistent cosmological modifications to the Einstein equations. 0806 1238
- [118] Nemanja Kaloper. Disformal inflation. Phys. Lett., B583:1-13, 2004. hep-ph/0312002.
- [119] Dimitrios Giannios. Spherically symmetric, static spacetimes in TeVeS. Phys. Rev., D71:103511, 2005. gr-qc/0502122.
- [120] J. T. Jebsen. Ark. Mat. Ast. Fys., 15(18), 1921.
- [121] G. D. Birkhoff. Relativity and Modern Physics. Harvard University Press, Cambridge (1923) 253p.
- [122] S. L. Dubovsky and S. M. Sibiryakov. Spontaneous breaking of Lorentz invariance, black holes and perpetuum mobile of the 2nd kind. Phys. Lett., B638:509-514, 2006. hep-th/0603158.
- [123] Christopher Eling, Brendan Z. Foster, Ted Jacobson, and Aron C. Wall. Lorentz violation and

- perpetual motion. Phys. Rev., D75:101502, 2007. hep-th/0702124.
- [124] De-Chang Dai, Reijiro Matsuo, and Glenn Starkman. Birkhoff's theorem fails to save MOND from non-local physics. 0811.1565.
- [125] Richard C. Tolman. Static solutions of Einstein's field equations for spheres of fluid. Phys. Rev., 55:364–373, 1939.
- [126] J. R. Oppenheimer and G. M. Volkoff. On Massive neutron cores. Phys. Rev., 55:374–381, 1939.
- [127] Xing-hua Jin and Xin-zhou Li. Oppenheimer-Volkoff Equation in Relativistic MOND Theory. gr-qc/0605046.
- [128] Paul D. Lasky, Hajime Sotani, and Dimitrios Giannios. Structure of Neutron Stars in Tensor-Vector-Scalar Theory. 0811.2006.
- [129] V. R. Pandharipande. Dense neutron matter with realistic interactions. Nucl. Phys., A174:641–656, 1971.
- [130] W. D. Arnett and R. L. Bowers. A Microscopic Interpretation of Neutron Star Structure. Astrophys. J. Suppl., 33:415–436, 1977.
- [131] Camille Bonvin, Chiara Caprini, and Ruth Durrer. Superluminal motion and closed signal curves. 0706.1538.
- [132] Simon DeDeo and Dimitrios Psaltis. Towards New Tests of Strong-field Gravity with Measurements of Surface Atomic Line Redshifts from Neutron Stars. Phys. Rev. Lett., 90:141101, 2003. astro-ph/0302095.
- [133] D. Sanwal, G. G. Pavlov, V. E. Zavlin, and M. A. Teter. Discovery of absorption features in the X-ray spectrum of an isolated neutron star. astro-ph/0206195.
- [134] J. Cottam, F. Paerels, and M. Mendez. Gravitationally redshifted absorption lines in the X-ray burst spectra of a neutron star. *Nature*, 420:51-54, 2002. astro-ph/0211126.
- [135] K. D. Kokkotas and Bernard F. Schutz. W-modes: A New family of normal modes of pulsating relativistic stars. Mon. Not. Roy. Astron. Soc., 255:119, 1992.
- [136] Kostas D. Kokkotas and Bernd G. Schmidt. Quasi-normal modes of stars and black holes. Living Rev. Rel., 2:2, 1999. gr-qc/9909058.
- [137] Simon DeDeo and Dimitrios Psaltis. Testing Strong-field Gravity with Quasi-Periodic Oscillations. astro-ph/0405067.
- [138] GianLuca Israel et al. Discovery of Rapid X-ray Oscillations in the Tail of the SGR 1806-20 Hyperflare. Astrophys. J., 628:L53–L56, 2005. astro-ph/0505255.
- [139] Tod E. Strohmayer and Anna L. Watts. Discovery of Fast X-ray Oscillations During the 1998 Giant Flare from SGR 1900+14. Astrophys. J., 632:L111-L114, 2005. astro-ph/0508206.
- [140] Tod E. Strohmayer and Anna L. Watts. The 2004 Hyperflare from SGR 1806-20: Further Evidence for Global Torsional Vibrations. Astrophys. J., 653:593-601, 2006. astroph/0608463.
- [141] Hajime Sotani, Kostas D. Kokkotas, and Nikolaos Stergioulas. Alfvén QPOs in Magnetars. Mon. Not. Roy. Astron. Soc., 385:L5, 2008. 0710.1113.
- [142] Hajime Sotani. Stellar Oscillations in Tensor-Vector-Scalar Theory. 0903.2424.
- [143] R. M. Wald. On identically closed forms locally constructed from a field. J. Math. Phys., 31:2378, 1990.
- [144] Michael D. Seifert and Robert M. Wald. A general variational principle for spherically symmetric perturbations in diffeomorphism covariant theories. *Phys. Rev.*, D75:084029, 2007. gr-qc/0612121.
- [145] Michael D. Seifert. Stability of spherically symmetric solutions in modified theories of gravity. Phys. Rev., D76:064002, 2007. gr-qc/0703060.
- [146] T. G. Zlosnik. Thesis title. PhD thesis, University of Oxford (2008).
- [147] C. Skordis and T. G. Zlosnik. In preparation.
- [148] Carlo R. Contaldi, Toby Wiseman, and Benjamin Withers. TeVeS gets caught on caustics. Phys. Rev., D78:044034, 2008. 0802.1215.
- [149] Jacob D. Bekenstein. Erratum: Relativistic gravitation theory for the MOND paradigm. Phys. Rev., D71:069901, 2005.
- [150] Takashi Tamaki. Post-Newtonian parameters in the tensor-vector-scalar theory. Phys. Rev., D77:124020, 2008. 0803.4309.
- [151] Brendan Z. Foster and Ted Jacobson. Post-Newtonian parameters and constraints on Einsteinaether theory. Phys. Rev., D73:064015, 2006. gr-qc/0509083.
- [152] E. Sagi. In preparation.
- [153] R. H. Sanders. Solar system constraints on multifield theories of modified dynamics. Mon. Not. Roy. Astron. Soc., 370:1519–1528, 2006. astro-ph/0602161.
- [154] Camille Bonvin, Ruth Durrer, Pedro G. Ferreira, Glenn Starkman, and Tom G. Zlosnik.

- Generalized Einstein-Aether theories and the Solar System. *Phys. Rev.*, D77:024037, 2008. 0707.3519.
- [155] Jacob Bekenstein and Joao Magueijo. MOND habitats within the solar system. Phys. Rev., D73:103513, 2006. astro-ph/0602266.
- [156] Hong-Sheng Zhao. An introduction to gravitational lensing in TeVeS gravity. astro-ph/0611777.
- [157] Mu-Chen Chiu, Chung-Ming Ko, and Yong Tian. Theoretical aspects of gravitational lensing in TeVes. Astrophys. J., 636:565–574, 2006. astro-ph/0507332.
- [158] J. A. Munoz et al. The CASTLES project. Astrophys. Space Sci., 263:51-54, 1999. astro-ph/9902131.
- [159] Da-Ming Chen and HongSheng Zhao. Strong gravitational lens probability in TeVeS. astro-ph/0606506.
- [160] Da-Ming Chen. Strong lensing probability in TeVeS theory. JCAP, 0801:006, 2008. 0712.1633.
- [161] M. Feix, C. Fedeli, and M. Bartelmann. Asymmetric Gravitational Lenses in TeVeS and Application to the Bullet Cluster. 0707.0790.
- [162] Garry W. Angus, Benoit Famaey, and HongSheng Zhao. Can MOND take a bullet? Analytical comparisons of three versions of MOND beyond spherical symmetry. Mon. Not. Roy. Astron. Soc., 371:138, 2006. astro-ph/0606216.
- [163] Dong Xu et al. Is Gravitational Lensing by Intercluster Filaments Always Negligible? Astrophys. J., 682:711–720, 2008. 0710.4935.
- [164] A. Mahdavi, H. y Hoekstra, A. y Babul, D. y Balam, and P. Capak. A Dark Core in Abell 520. 0706.3048.
- [165] Huan Yuan Shan, Martin Feix, Benoit Famaey, and HongSheng Zhao. An analytic model for non-spherical lenses in covariant MOdified Newtonian Dynamics. Mon. Not. Roy. Astron. Soc., 387:1303-1312, 2008. 0804.2668.
- [166] Hong-Sheng Zhao, David J. Bacon, Andy N. Taylor, and Keith Horne. Testing Bekenstein's Relativistic MOND gravity with Gravitational Lensing. Mon. Not. Roy. Astron. Soc., 368:171–186, 2006. astro-ph/0509590.
- [167] M. C. Chiu, Y. Tian, and C. M. Ko. Necessity of Dark Matter in Modified Newtonian Dynamics within Galactic Scales? - Testing the Covariant MOND in Elliptical Lenses. 0812.5011.
- [168] Nick E. Mavromatos, Mairi Sakellariadou, and Muhammad Furqaan Yusaf. Can TeVeS avoid Dark Matter on galactic scales? 0901.3932.
- [169] Joshua W. Elliott, Guy D. Moore, and Horace Stoica. Constraining the new aether: Gravitational Cherenkov radiation. JHEP, 08:066, 2005. hep-ph/0505211.
- [170] Jean-Philippe Bruneton. On causality and superluminal behavior in classical field theories. Applications to k-essence theories and MOND-like theories of gravity. *Phys. Rev.*, D75:085013, 2007. gr-qc/0607055.
- [171] E. O. Kahya and R. P. Woodard. A generic test of modified gravity models which emulate dark matter. Phys. Lett., B652:213–216, 2007. 0705.0153.
- [172] E. O. Kahya. A Decisive test to confirm or rule out existence of dark matter using gravitational wave observations. Class. Quant. Grav., 25:184008, 2008. 0801.1984.
- [173] S. Desai, E. O. Kahya, and R. P. Woodard. Reduced time delay for gravitational waves with dark matter emulators. journal =.
- [174] R. H. Sanders. A tensor-vector-scalar framework for modified dynamics and cosmic dark matter. Mon. Not. Roy. Astron. Soc., 363:459, 2005. astro-ph/0502222.
- [175] T. G. Zlosnik, P. G. Ferreira, and Glenn D. Starkman. The vector-tensor nature of Bekenstein's relativistic theory of modified gravity. *Phys. Rev.*, D74:044037, 2006. gr-qc/0606039.
- [176] T. G Zlosnik, P. G. Ferreira, and G. D. Starkman. On the growth of structure in theories with a dynamical preferred frame. Phys. Rev., D77:084010, 2008. 0711.0520.
- [177] Baojiu Li, David Fonseca Mota, and John D. Barrow. Detecting a Lorentz-Violating Field in Cosmology. Phys. Rev., D77:024032, 2008. 0709.4581.
- [178] Joseph A. Zuntz, P. G. Ferreira, and T. G. Zlosnik. Constraining Lorentz violation with cosmology. 0808.1824.
- [179] T. G. Zlosnik and J. A. Zuntz. In preparation.
- [180] De-Chang Dai, Reijiro Matsuo, and Glenn Starkman. Gravitational Lenses in Generalized Einstein-Aether theory: the Bullet Cluster. Phys. Rev., D78:104004, 2008. 0806.4319.
- [181] C. Skordis. In preparation.
- [182] Anaelle Halle and HongSheng Zhao. Perturbations In A Non-Uniform Dark Energy Fluid: Equations Reveal Effects of Modified Gravity and Dark Matter. 0711.0958.
- [183] HongSheng Zhao. Coincidences of Dark Energy with Dark Matter Clues for a Simple Alternative? 0710.3616.

[184] Brendan Z. Foster. Metric redefinitions in Einstein-aether theory. Phys. Rev., D72:044017, 2005. gr-qc/0502066.

Appendix A. Alternative TeVeS conventions

In the original TeVeS article [37], Bekenstein uses a "tilde" for the matter-frame (what he calls the physical) metric, and no "tilde" for the Einstein-frame (what he calls geometric) metric. Furthermore the symbol \mathcal{U}_a is used for the vector field rather than A_a . The scalar field action, and field equations contain an auxiliary scalar field σ , related to μ by $\mu = 8\pi G \sigma^2$ (note that Bekenstein also uses a field μ which is not the same as the one in this review). Finally the scalar field free function is given as $F(kG\sigma^2)$ related to $V(\mu)$ as

$$V(\mu) = \frac{1}{16\pi\ell^2} \mu^2 F = \frac{4\pi G^2}{\ell^2} \sigma^4 F \tag{A.1}$$

Appendix B. Alternative and classically equivalent actions for a general class of TeVeS theories

Appendix B.1. The aquadratic kinetic term action

Since μ is a function of kinetic terms of ϕ , we can simply replace the scalar field action with

$$S = -\frac{1}{16\pi G\ell_B^2} \int d^4x \sqrt{-\tilde{g}} f(X)$$
 (B.1)

where

$$X = \ell_B^2 (\tilde{g}^{ab} + \beta A^a A^b) \nabla_a \phi \nabla_b \phi. \tag{B.2}$$

This is akin to the k-essence/k-inflation actions considered elsewhere [80, 81], but with an additional coupling of the vector field to the scalar. One can then simply transform between this and the action in the main part of the text via

$$\mu = \frac{df}{dX} \tag{B.3}$$

and

$$f = \mu X + \ell_B^2 V \tag{B.4}$$

This action may be more intuitive, in making contact with the MOND limit in the quasistatic case.

Appendix B.2. The single metric frame

Zlosnik, Ferreira and Starkman have shown how to write TeVeS theory completely in the universally coupled frame [175]. This is possible by defining a new vector field $B_a = A_a$ such that $B^a = g^{ab}B_b$. The magnitude of this vector field with respect to the universally coupled metric is related to the scalar field ϕ as

$$B^2 = g^{ab} B_a B_b = -e^{-2\phi} (B.5)$$

It is thus possible to eliminate ϕ from the action in terms of B^2 and eventually B_a . At the same time, it is also possible to eliminate the metric \tilde{g}_{ab} in terms of g_{ab} and g_a as

$$\tilde{g}_{ab} = -\frac{1}{R^2}g_{ab} + (\frac{1}{R^4} - 1)B_aB_b \tag{B.6}$$

and similarly for \tilde{g}^{ab} . Since the above relations are algebraic, they can be used in the action. The final step consists of changing connection from $\tilde{\nabla}_a$ (metric compatible with \tilde{g}_{ab}) to ∇_a (metric compatible with g_{ab}).

The same procedure can also be performed for the generalized TeVeS theory [78] (see section 7.3). A very lengthly and tedious calculation gives the action in the universally coupled frame as

$$S = \int d^4x \sqrt{-g} \left[R - K^{abcd} \nabla_a B_b \nabla_c B_d + \frac{1}{B^2} V(\mu) \right] + S_m[g] \quad (B.7)$$

where the tensor K^{abcd} is given by

$$K^{abcd} = d_1 g^{ac} g^{bd} + d_2 g^{ab} g^{cd} + d_3 g^{ad} g^{bc} + d_4 B^a B^c g^{bd}$$

$$+ \frac{1}{2} d_5 (g^{ad} B^b B^c + g^{bc} B^a B^d) + d_6 g^{ac} B^b B^d$$

$$+ \frac{1}{2} d_7 (g^{ab} B^c B^d + g^{cd} B^a B^b) + d_8 B^a B^b B^c B^d$$
(B.8)

with the coefficients being

$$d_1 = \frac{1 + c_3 - c_1}{2}B^2 - \frac{1}{B^2} + \frac{1 - c_{13}}{2B^6}$$
(B.9)

$$d_2 = \frac{1}{B^2} - \frac{1+c_2}{B^6} \tag{B.10}$$

$$d_3 = \frac{c_1 - c_3 - 1}{2}B^2 + \frac{1 - c_{13}}{2B^6}$$
(B.11)

$$d_4 = \frac{c_1 - c_3 - 1}{2} + \frac{c_4 - c_1 + 1}{B^4} + \frac{c_{13} - 1}{2B^8}$$
(B.12)

$$d_5 = 1 + c_3 - c_1 + \frac{2(c_1 - c_4 - 2)}{B^4} + \frac{c_{13} - 1}{B^8}$$
(B.13)

$$d_6 = \frac{c_1 - c_3 - 1}{2} + \frac{3 + c_4 - c_1 + \mu}{B^4} + \frac{c_{13} - 1}{2B^8}$$
(B.14)

$$d_7 = -\frac{2}{B^4} + \frac{2(3 + c_{13} + 4c_2)}{B^8} \tag{B.15}$$

$$d_8 = \frac{2 - \mu}{B^6} + \frac{(1 - \beta)\mu - 6c_{13} - 16c_2 - 10}{B^{10}}$$
(B.16)

and where the constants c_i appear in the vector field action (see section 7.3). In the case of the original TeVeS, the above coefficients differ from those in [175] which are incorrect. The correct coefficients can also be found in Zlosnik's PhD thesis [146].

Notice that the time-like constraint is now absent, since it has been used to eliminate the scalar field. The scalar field is absorbed into the vector field, which now has four, rather than three independent degrees of freedom. The fact that the vector field is timelike is now imposed dynamically, due to the presence of inverse powers of its modulus B^aB_a , i.e. the vector field can never be null. It seems that this action describes two disconnected sectors, one where the vector field is timelike and one where it is spacelike, and any transitions between them are forbidden. However the spacelike section, when expanded around the background value $B^2 = 1$, leads to wrong-sign kinetic terms. Therefore only the timelike sector is the physical one, which is precisely TeVeS theory.

Appendix B.3. The diagonal frame

In the diagonal frame, one performs a yet another disformal transformation to a new metric \hat{g}_{ab} , as well as a field redefinition of ϕ and A_a . We shall consider a transformation of a more general version of TeVeS. The vector field action is generalized to an Einstein-Æther action with coefficients c_i (see section 7.3) as in [78]. We also generalize the scalar field action to

$$S_{\phi} = -\frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} \left[\mu(\tilde{g}^{ab} + \beta A^a A^b) \nabla_a \phi \nabla_b \phi + V(\mu) \right] \quad (B.17)$$

where the new parameter β is equal to $\beta = -1$ in the original TeVeS theory, such that $\beta < 1$ to preserve the metric signature.

The new fields are defined by

$$\hat{g}^{ab} = \frac{1}{\sqrt{1-\beta}} [\tilde{g}^{ab} + \beta A^a A^b], \tag{B.18}$$

$$\hat{g}_{ab} = \sqrt{1 - \beta} \tilde{g}_{ab} - \frac{\beta}{\sqrt{1 - \beta}} A_a A_b, \tag{B.19}$$

$$\chi = \phi + \frac{1}{4}\ln(1-\beta),$$
(B.20)

$$\hat{A}_a = (1 - \beta)^{-1/4} A_a, \tag{B.21}$$

$$\hat{A}^a = (1 - \beta)^{1/4} A^a. \tag{B.22}$$

The new vector field \hat{A}_a remains unit-timelike with respect to the new metric: $\hat{A}_a\hat{A}^a=\hat{g}^{ab}\hat{A}_a\hat{A}_b=-1$. Notice that for $\beta=-1$, the inverse metric transformation takes the same form as the one appearing in the scalar field action in TeVeS. Thus with this transformation, the scalar field completely decouples from the vector field in this frame. With respect to this metric, as the name implies, the kinetic terms of the new metric, the scalar field and the new vector field become diagonalized. Disregarding the scalar field, this transformation has been considered by Foster [184], in the context of Einstein-Æther theory.

Under this transformation, the TeVeS action becomes

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-\hat{g}} \left[\hat{R} - \hat{K}^{abcd} \hat{\nabla}_a \hat{A}_b \, \hat{\nabla}_c \hat{A}_d + \lambda (\hat{A}^a \hat{A}_a + 1) \right.$$
$$\left. - \mu \hat{g}^{ab} \nabla_a \chi \nabla_b \chi - \hat{V}(\mu) \right] + S_m[g] \tag{B.23}$$

where the Lagrange multiplier has been appropriately rescaled, $\hat{V} = V/\sqrt{1-\beta}$ and where

$$\hat{K}^{abcd} = \hat{c}_1 \hat{g}^{ac} \hat{g}^{bd} + \hat{c}_2 \hat{g}^{ab} \hat{g}^{cd} + \hat{c}_3 \hat{g}^{ad} \hat{g}^{bc} + \hat{c}_4 \hat{A}^a \hat{A}^c \hat{g}^{bd}.$$
 (B.24)

The new coefficients \hat{c}_i are related to the old ones c_i via

$$\hat{c}_{1} = \frac{(2 - 2\beta + \beta^{2})c_{1} + \beta(2 - \beta)c_{3} - \beta^{2}}{2(1 - \beta)},$$

$$\hat{c}_{2} = \frac{c_{2} + \beta}{1 - \beta},$$

$$\hat{c}_{3} = \frac{(2 - 2\beta + \beta^{2})c_{3} + \beta(2 - \beta)(c_{1} - 1)}{2(1 - \beta)},$$

$$\hat{c}_{4} = c_{4} + \frac{\beta^{2}(c_{1} - 1) + \beta(2 - \beta)c_{3}}{2(1 - \beta)}$$
(B.25)

Variation of the action proceeds by noting that the physical metric is now given in terms of \hat{g}_{ab} as

$$g_{ab} = e^{-2\chi} \hat{g}_{ab} - 2\sinh(2\chi)\hat{A}_a \hat{A}_b \tag{B.26}$$

which is the same as the standard TeVeS transformation. Note that in the case of the original TeVeS theory, for which $\beta=-1,\,\tilde{c}_2=\tilde{c}_4=0$ and $\tilde{c}_1=K=-\tilde{c}_3$ we get

$$\hat{c}_1 = 2K - \frac{1}{4} \qquad \hat{c}_2 = -\frac{1}{2}
\hat{c}_3 = -2K + \frac{3}{4} \qquad \hat{c}_4 = K - \frac{1}{4}$$
(B.27)

$$\hat{c}_3 = -2K + \frac{3}{4}$$
 $\hat{c}_4 = K - \frac{1}{4}$ (B.28)

while in terms of the linear combinations K_t , κ_d , K_F and R_K as in [78] (see section 7.3) we get

$$K_t = K \kappa_d = 0 (B.29)$$

$$K_t = K$$
 $\kappa_d = 0$ (B.29)
 $K_F = \frac{1}{2}$ $R_K = 1$ (B.30)

Thus this transformation does not introduce any damping term κ_d .